Math 8245 **Homework 3** **PJW**

Date due: October 18, 2010. Either hand it to me in class or put it in my mailbox by 3:30.

1. Use GAP to show that

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (bc)^3 = (ca)^5 = 1\rangle \cong A_5 \times C_2.$$

2. The generalized quaternion group of order 2^n has a presentation

$$\langle a, b \mid a^{2^{n-1}} = 1,\, b^2 = a^{2^{n-2}},\, bab^{-1} = a^{-1}\rangle.$$

   Use GAP to investigate the generalized quaternion group of order 32. Get a list of the orders of the elements. Compute the derived subgroup and the center. Draw a picture of the lattice of subgroups of this group. What is the minimum degree of a faithful permutation representation of this group?

3. (a) Show that every homomorphism of G-sets $\Omega \to \Psi$ where Ψ is transitive is necessarily an epimorphism.
   (b) Let Ψ be a transitive G-set and assume for this part of the question that G is finite. Show that every G-set mapping $\Psi \to \Psi$ is a bijection.
   (c) Let H and K be subgroups of G. Show that every homomorphism of G-sets $G/H \to G/K$ is a composite $G/H \to G/J \to G/K$ where $H \leq J$, J is conjugate to K, and the mapping $G/H \to G/J$ is $xH \mapsto xJ$.

4. (Question 11.4 from the handout) A group G is *injective* $\Leftrightarrow$ whenever we are given a subgroup A of a group B and a homomorphism $f : A \to G$ there exists a homomorphism $g : B \to G$ so that the restriction of g to A is f. Prove that injective groups have order 1. [Hint (D.L. Johnson): let A be free on $\{a, b\}$ and let $B = A \rtimes \langle c\rangle$ where c has order 2 and $cac^{-1} = b$, $cbc^{-1} = a$.]

5. (Question 11.5 from the handout) Let $X = \{x_k \mid k \in K\}$ and let $Y \subseteq X$. If F is free on X and H is the normal subgroup generated by Y, show that F/H is free.

6. (Question 11.6 from the handout) Show that a free group F on $\{x, y\}$ has an automorphism f with $f(f(a)) = a$ for all $a \in F$ and with the further property that $f(a) = a$ if and only if $a = 1$.

Extra Questions: do not hand in

7. Use GAP to show that $SL(2,5)$ has a normal subgroup of order 2 such that the quotient is isomorphic to A_5. Show that $SL(2,5)$ has no subgroup isomorphic to A_5. Identify the Sylow 2-subgroups of $SL(2,5)$.

8. Use GAP to investigate the groups

$$\langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^2 = (bc)^3 = (ac)^4 = 1\rangle$$

and

$$\langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^2 = (bc)^3 = (ac)^3 = 1\rangle$$

In each case identify the quotient by the center $G/Z(G)$ and determine whether or not $G = Z(G) \times H$ for some subgroup H.

9. Let F be a free group of rank 2. Show that it is possible to find a set of three elements which generate F, no two of which generate F.

10. I can't see how to do the following; can you? I suppose it is true.
Let F be a free group of rank n and let X be a subset of n which generates F. Show that X generates F freely.