Notation: $M^\lambda$ is the permutation module with the $\lambda$-tabloids as a basis. $S_F(n,r) = \text{End}_{F S_n}(E^\otimes r)$ where $E$ is a vector space of dimension $n$ over $F$.

1. Find a basis for the space of homomorphisms $\text{Hom}_{F S_5}(M^{(3,2)}, M^{(2,1,1,1)})$. For each element $\theta$ in your basis, compute the effect of $\theta$ on the tabloid

\[
\begin{array}{cccc}
1 & 2 & 3 \\
4 & 5 \\
\end{array}
\]

2. (a) Let $F$ be an infinite field, and suppose that $\rho : GL_1(F) \to GL_1(F)$ is a 1-dimensional polynomial representation of $GL_1(F)$. Show that $\rho$ has the form $\rho(\lambda) = \lambda^n$ for some $n$. [Hint: Allowing $\lambda$ to vary, compare $\rho(\lambda)^m$ and $\rho(\lambda^m)$, and use the fact that a polynomial which vanishes identically must be zero.]

(b) Exhibit a 1-dimensional representation of $GL_1(\mathbb{C})$ which is not rational (i.e. not of the form $\rho(\lambda) = f(\lambda)/g(\lambda)$ for polynomials $f$ and $g$).

3. (a) Show that every subrepresentation of a polynomial representation is polynomial. [You may use without proof the fact that if $\rho : GL_n(F) \to GL(V)$ is given by matrices $\rho_{i,j}(g)$ with respect to one basis of $V$, and matrices $\sigma_{i,j}(g)$ with respect to another basis, then the $\rho_{i,j}$ are polynomial functions (homogeneous of degree $r$) if and only if the $\sigma_{i,j}$ are.]

(b) Show that if $V_1, V_2$ are invariant subspaces of a representation $V$ of $GL_n(F)$, and that $V_1$ and $V_2$ are polynomial (homogeneous of degree $r$) then $V_1 + V_2$ is polynomial (homogeneous of degree $r$). Deduce that each finite dimensional representation $V$ of $GL_n(F)$ has a unique largest subrepresentation which is polynomial, and also for each $r$ a unique largest subrepresentation which is polynomial of degree $r$.

4. Show by example that the homomorphism $FGL(E) \to S_F(n,r)$ given by the representation of $GL(E)$ on $E^\otimes r$ need not be surjective if the field $F$ is not infinite.

5. Let $H$ be the subspace of $E^\otimes r$ spanned by the elements

\[e_{i_1} \otimes \cdots \otimes e_{i_r} - e_{i(1)\pi} \otimes \cdots \otimes e_{i(r)\pi}\]

as $\pi$ ranges over the elements of $S_r$. The $r$th symmetric power of $E$ is defined to be

\[S^r(E) := E^\otimes r / H.\]

The image of a tensor $e_{i_1} \otimes \cdots \otimes e_{i_r}$ in $S^r(E)$ may be written as a monomial $e_{i_1} \cdots e_{i_r}$. The degree $r$ symmetric tensors $ST^r(E)$ are defined to be the fixed points of $S_r$ in its action on $E^\otimes r$.

(a) Show that $ST^r(E)$ is a $FGL(E)$-submodule of $E^\otimes r$. 


(b) Show that $ST^r(E)$ is spanned by tensors of the form

$$\sum_{j \in \iota S_r} e_j$$

where the sum is over the $S_r$-orbit of multi-indices containing $\iota$.

(c) When $F$ has characteristic 0 or $p > r$ show that $E^{\otimes r} = ST^r(E) \oplus H$. Deduce that $ST^r(E) \cong S^r(E)$ as $GL(E)$-modules.

(d) Let $F$ be a field of characteristic 2. In the particular case when $\dim E = 2$ and $r = 2$ show that $ST^2(E)$ has a 1-dimensional $FGL(E)$-submodule (hint: it is the space of skew-symmetric tensors) but that $S^2(E)$ has no 1-dimensional $FGL(E)$-submodule. (Hint: consider the action of matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ on $S^2(E)$.)

Deduce that $S^2(E)$ is not isomorphic to $ST^2(E)$ as $FGL(E)$-modules.

[In fact $S^r(E)$ is isomorphic to the dual (in a certain sense) of $ST^r(E)$. These two modules have the same composition factors - $\Lambda^2(E)$ and $E$ in case $r = 2$.]