The blocks of the Steiner system $S(5, 8, 24)$ are called *octads.*

PROPOSITION (The Leech triangle). *In the Steiner system $S(5, 8, 24)$ let $O = \{x_1, \ldots, x_8\}$ be an octad, and put $O_i = \{x_1, \ldots, x_i\}$ if $i \leq 8$. Let a_{ij} be the number of octads which intersect O_i in exactly O_j with $j \leq i$. Then $a_{ii} = \lambda_i$ and $a_{ij} = a_{i+1,j} + a_{i+1,j+1}$. These numbers form a triangle in which each term is the sum of the two below it.*

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&&759\\
&&&&&&&506&&253\\
&&&&&&330&&176&&77\\
&&&&&210&&120&&56&&21\\
&&&&130&&80&&40&&16&&5\\
&&&78&&52&&28&&12&&4&&1\\
&&46&&32&&20&&8&&4&&0&&1\\
&30&&16&&16&&4&&4&&0&&0&&1\\
30&&0&&16&&0&&4&&0&&0&&0&&1
\end{array}
$$

COROLLARY. *If O_1 and O_2 are octads then $|O_1 \cap O_2| = 0, 2$ or 4.*

We define a vector space $\mathcal{C}$ over $\mathbb{F}_2$. For each finite set X we denote by $\mathcal{P}(X)$ the set of subsets of X. There is a bijection

$$
\begin{aligned}
\mathcal{P}(X) &\leftrightarrow \text{elements of } \mathbb{F}_2^{|X|}\\
A &\leftrightarrow \chi_A, \text{ the characteristic function.}
\end{aligned}
$$

Under this bijection $\chi_A + \chi_B = \chi_{A+B}$ where $A + B = (A \cup B) - (A \cap B)$ is the *symmetric difference* of A and B. We now define $\mathcal{C}$ to be the subspace of $\mathcal{P}(X)$ spanned by the octads, where now X is the set of 24 points of $S(5, 8, 24)$. This subspace is called the *extended binary Golay code.*

PROPOSITION. *$X \in \mathcal{C}$.*

Proof. $\lambda_1 = 253$ is odd, so the sum of all octads is X since $\chi_{A_1+\cdots+A_t}(x) = 1$ if and only if x lies in an odd number of the A_i. □

PROPOSITION. *if $Y \in \mathcal{C}$ and O is an octad then $|O \cap Y|$ is even.*

Proof. We use induction on the number of terms in an expression for Y as a sum of octads. When Y is an octad the result is true from the Leech triangle, so the induction starts. If Y_1 and Y_2 lie in $\mathcal{C}$ and have even intersectiions with O then $|O \cap (Y_1 + Y_2)| = |O \cap Y_1| + |O \cap Y_2| - 2|O \cap Y_1 \cap Y_2|$ and this is even. □

PROPOSITION. *Every 8-element set in $\mathcal{C}$ is an octad.*

Proof. Let $Y \in \mathcal{C}$ have size 8. Any 5-element subset of Y is contained in a unique octad O, and if Y is not an octad then $|Y \cap O| = 6$. It follows that sets of the form $O \cap Y$ of size 6 where O is an octad form a Steiner system with parameters $S(5, 6, 8)$. The number of blocks in such a Steiner system is $(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4)/(6 \cdot 5 \cdot 4 \cdot 3 \cdot 2) = 28/3$ which is not an integer, so no such Steiner system can exist. Hence Y is an octad. $\square$

COROLLARY. *If O_1 and O_2 are octads with $|O_1 \cap O_2| = 4$ then $O_1 + O_2$ is an octad.*

COROLLARY. *If O_1 and O_2 are octads with $|O_1 \cap O_2| = \emptyset$ then $X - (O_1 \cup O_2)$ is an octad.*

PROPOSITION. *Let S_1 be any 4-element subset of X. The five octads containing S_1 have the form $S_1 \cup S_2$, $S_1 \cup S_3, \ldots, S_1 \cup S_6$ where the S_i are 4-element subsets. These subsets have the properties that $S_1 \cup \cdots \cup S_6 = X$, and for each pair $i \neq j$, $S_i \cup S_j$ is an octad.*

Such a configuration of six 4-element subsets is called a *sextet*.

Proof. The fact that there are five octads containing a four element set follows from the Leech triangle. Their union is the whole of X since any point of X may be adjoined to the four to give a five element sets which is contained in an octad. The union of any pair of the S_i is an octad since it lies in $\mathcal{C}$ and has size 8. $\square$

Suppose two octads O_1 and O_2 have $|O_1 \cap O_2| = 2$. The 12-element set $O_1 + O_2$ is called a *dodecad* (or sometimes an *umbral dodecad*).

PROPOSITION. *A dodecad does not contain any octad.*

Proof. Suppose we have octads $O_3 \subset O_1 + O_2$ where $|O_1 \cap O_2| = 2$. Then O_3 is distinct from O_1 and O_2 so $|O_1 \cap O_3| \leq 4$ and $|O_2 \cap O_3| \leq 4$ (since if an intersection had size 5 or larger the octads would be the same). Thus $|O_1 \cap O_3| = |O_2 \cap O_3| = 4$ and so $O_1 + O_3$ is an octad, and it contains 2 points which do not lie in O_2. Now $|(O_1 + O_3) \cap O_2| = 6$, which is a contradiction. $\square$

COROLLARY. *Let D be a dodecad. The subsets $O \cap D$ of size 6 with O an octad form a Steiner system $S(5, 6, 12)$.*

The special sets $O \cap D$ of size 6 are called *hexads*.

PROPOSITION. *The Steiner System $S(5,6,12)$ has a Leech triangle:*

$$\begin{array}{ccccccccccccc}
 & & & & & & 132 & & & & & & \\
 & & & & & 66 & & 66 & & & & & \\
 & & & & 30 & & 36 & & 30 & & & & \\
 & & & 12 & & 18 & & 18 & & 12 & & & \\
 & & 4 & & 8 & & 10 & & 8 & & 4 & & \\
 & 1 & & 3 & & 5 & & 5 & & 3 & & 1 & \\
1 & & 0 & & 3 & & 2 & & 3 & & 0 & & 1
\end{array}$$

COROLLARY. *The complement of a hexad in D is a hexad.*

Proof. This is indicated by the 1 at the bottom left corner. □

LEMMA. *The complement $X - D$ of a dodecad D is a dodecad.*

Proof. Let $D = O_1 + O_2$ with $|O_1 \cap O_2| = 2$, and let O_3 be any octad disjoint from O_1. Let O_4 be the complement $X - (O_1 \cup O_3)$. Then $O_2 \cap O_3 \subseteq O_2 - O_1$ which has size 6, so $|O_2 \cap O_3| = 0, 2$ or 4. Similarly for $O_2 \cap O_4$ and without loss of generality $|O_2 \cap O_3| = 2$ and $|O_2 \cap O_4| = 4$. Now $O_2 + O_4$ is an octad and $X - D = (O_2 + O_4) + O_3$. □

LEMMA. *Let D be a dodecad and O_1 an octad so that $|O_1 \cap D| = 6$. Then $O_2 = O_1 + D$ is an octad such that $D = O_1 + O_2$. The sets $O_1 \cap D$ and $O_2 \cap D$ are complementary hexads.*

Proof. The set $O_2 = O_1 + D$ has size 8 and lies in $\mathcal{C}$, so is an octad. Hence $O_2 + O_1 = O_1 + D + O_1 = D$. □

LEMMA. *The number of dodecads is 2576.*

Proof. Suppose that D is a dodecad and suppose that $D = H_1 \cup H_2$ is a decomposition into complementary hexads, where $H_i = D \cap O_i$. The pair of points $O_1 \cap O_2$ completely determine this decomposition, because if also $D = O_1' + O_2'$ is a different decomposition with $O_1 \cap O_2 = O_1' \cap O_2'$ then both $O_1' \cap O_1$ and $O_1' \cap O_2$ have size at most 4, and hence $|O_1' \cap D| \leq 4$, a contradiction.

Now D contains 66 pairs of complementary hexads. $X - D$ contains $\binom{12}{2} = 66$ pairs of points. Each pair in $X - D$ is associated with at most one pair of hexads. Therefore pairs of complementary hexads in D biject with pairs of points in $X - D$.

The number of unordered pairs of octads O_1, O_2 such that $|O_1 \cap O_2| = 2$ is

$$\frac{759 \times \binom{8}{2} \times 16}{2}$$

the 16 coming from the Leech triangle. The number of decompositions of $O_1 + O_2$ into such a pair is 66. Therefore the number of dodecads is

$$\frac{759 \times \binom{8}{2} \times 16}{2 \times 66} = 2576.$$

PROPOSITION. *The sets in $\mathcal{C}$ are the empty set, octads, dodecads, complements of octads and X.*

Proof. We show that these sets are preserved under symmetric difference with octads. We have already seen that the symmetric difference of two octads is of the specified form.

Consider now $O + D$ were O is an octad and D is a dodecad. In this case $|O \cap D|$ is even and less than 8, and similarly with $|O \cap (X - D)|$, so $|O \cap D| = 2, 4, 6$ and $|O \cap (X - D)| = 6, 4, 2$, respectively. The case of intersections of size 6 has just been considered.

Suppose that $|D \cap O| = 4$, so $|D + O| = 12$. Let $H_1 = D \cap O_1$ be a hexad containing $O \cap D$, and let $H_2 = O_2 \cap D$ be the complementary hexad. Now $D = O_1 + O_2$ and $D + O = O_1 + O_2 + O = (O + O_1) + O_2$. Here $O + O_1$ is an octad, and so $D + O$ is of the specified form.

Finally the complement of any octad may be written as a union of octads $O_1 + O_2$, and now the symmetric difference with an octad O reduces to the previous cases on considering $(O + O_1) + O_2$. □

COROLLARY. $\dim_{\mathbb{F}_2} \mathcal{C} = 12$.

Proof. The number of vectors in $\mathcal{C}$ is $1 + 759 + 2576 + 759 + 1 = 4096 = 2^{12}$. □

The rows of the following matrix form a basis for a subspace of $\mathbb{F}_2^{24}$ which after relabeling the columns is $\mathcal{C}$:

$$\begin{pmatrix}
1&1&0&1&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&1\\
0&1&1&0&1&0&0&1&0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&1\\
0&0&1&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&1&1&0&1&0&1\\
0&0&0&1&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&1\\
0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&1&1&1&0&1&0&0&0&1\\
0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&1&0&1&1&0&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&1&0&1&0&1&0&0&1&1&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&1&0&0&0&1&1&0&1&1\\
0&0&0&1&0&1&1&1&0&0&0&1&0&1&1&1&0&0&0&1&0&1&1&1\\
0&0&1&0&1&1&0&1&0&0&1&0&1&1&0&1&0&0&1&0&1&1&0&1\\
0&1&0&1&1&0&0&1&0&1&0&1&1&0&0&1&0&1&0&1&1&0&0&1\\
1&0&1&1&0&0&0&1&1&0&1&1&0&0&0&1&1&0&1&1&0&0&0&1
\end{pmatrix}$$