Math 8211 **Commutative and Homological Algebra I** **Fall 2021**

Homework Assignment 3 Due Saturday 11/27/2021, uploaded to Gradescope.

1. Let k be a field. A monomial ideal I of $k[x_0, \ldots, x_r]$ is defined to be an ideal generated by monomials. Show that such an ideal. I has a basis over k consisting of the monomials it contains, and that the monomials not in I span a subspace W of $k[x_0, \ldots, x_r]$ so that $k[x_0, \ldots, x_r] = I \oplus W$.

Solution. If I is generated by monomials $m_1, m_2, \ldots$ then the typical element of I can be written $\sum_{i \in J} f_i m_i$ where J is some finite subset of the indices and the f_i are polynomials. Such an expression lies in the span of all the monomials $u_t m_i$ where u_t ranges over all monomials (because each f_i lies in the span of such u_t), and these monomials lie in I. They are also independent, because distinct monomials sets are independent, so they form a basis for I. Evidently the monomials in I taken together with the monomials not in I form a basis for the whole of $k[x_0, \ldots, x_r]$, and dividing up the basis into two subsets in this way gives the required decomposition $I \oplus W$.

2. (Eisenbud 3.6) Eisenbud characterizes prime monomial ideals as ideals 'generated by subsets of the variables,' irreducible monomial ideals as ideals 'generated by powers of some of the variables,' radical monomial ideals as 'ideals generated by square-free monomials,' and primary monomial ideals as ideals 'containing a power of each of a certain subset of the variables, and generated by elements involving no further variables.'
(a) Prove Eisenbud's characterization of prime monomial ideals.
(b) Prove Eisenbud's characterization of irreducible monomial ideals.
(c) Prove Eisenbud's characterization of radical monomial ideals.
(d) Prove Eisenbud's characterization of primary monomial ideals.

Solution Let I be a monomial ideal.
(a) Suppose I is prime. If I contains a monomial $x_{i_1} \cdots x_{i_t}$ then I contains some x_{i_j} because it is prime, and the monomial lies in (x_{i_j}), so I is generated by variables x_{i_j}. Conversely, if I is generated by $\{x_j \mid j \in J\}$ where J is some subset of $\{0, \ldots, r\}$ then $k[x_0, \ldots, x_r]/I \cong k[\{x_j \mid j \notin J\}]$ and this is a domain, so I is prime.
(b) If I is a monomial irreducible ideal and $m_1 m_2$ is a monomial in I where m_1 and m_2 have no common factors then $I = (I + (m_1)) \cap (I + (m_2))$ is the intersection of two ideals, so at least one of them must be I, and $m_1 m_2$ cannot be part of a minimal generating set for I. Thus a minimal generating set for I consists of powers of variables. Conversely, suppose I is generated by powers of variables and that $I = J \cap K$ with $J \neq I \neq K$. We can find $x_j^m a \in J$ and $x_k^n b \in K$ where the variable x_j does not divide a and x_k does not divide b with $x_j^m \notin I$ and $x_k^n \notin I$. If $j \neq k$ then $x_j^m a x_k^n b \in (J \cap K) - I$, and if $j = k$ then $x_j^t ab \in (J \cap K) - I$ where t is the maximum of $\{m, n\}$. This is a contradiction, so I is irreducible.

(c) If I is a radical monomial ideal and $x_{i_1}^{n_1} \cdots x_{i_r}^{n_r} \in I$ then $x_{i_1} \cdots x_{i_r} \in I$, so I is generated by square free monomials. Conversely if I is generated by square free monomials and $f^n \in I$ then, because a power of every monomial in f appears in f^n, all the monomials in f lie in I, so $f \in I$.
(d) Suppose I contains powers of $x_{i_1}, \ldots, x_{i_t}$ and is generated by monomials in these variables and no others. Then the radical of I is $(x_{i_1}, \ldots, x_{i_t})$, which is thus the only minimal prime ideal containing I and is an associated prime of I. If u annihilates an element of R/I then it annihilates an element of $R/(x_{i_1}, \ldots, x_{i_t})$, which is a domain, so $u \in (x_{i_1}, \ldots, x_{i_t})$. Thus $(x_{i_1}, \ldots, x_{i_t})$ is the only associated prime of I, which is primary. Conversely, if I is primary for some prime, which by part (a) is an ideal generated by a subset of the variables, and this is the only minimal prime containing I (because all minimal primes are associated primes). The radical of I is thus this prime, and some power of it is contained in I. This means that I contains powers of those variables, and is generated by monomials involving only those variables.

3. (Eisenbud 3.8) Find an algorithm for computing an irreducible decomposition, and thus a primary decomposition, of a monomial ideal.

Solution. Given a set of monomial generators for the ideal I, for each such generator write it as a product of powers of the variables. Take the irreducible ideals generated by all lists of such powers of the variables, one power taken from each generator. This gives an irreducible decomposition of I.

4. Find an irreducible decomposition, and also two different minimal primary decompositions of the ideal (x^4y, x^2y^2, y^3) in $k[x, y]$.

Solution. Applying the result of problem 3 we construct the irreducible ideals (x^4, x^2, y^3), (x^4, y^2, y^3), (y, x^2, y^3) and (y, y^2, y^3). These ideals equal (x^2, y^3), (x^4, y^2), (y, x^2) and (y). The intersection of these is an irreducible decomposition of (x^4y, x^2y^2, y^3), but because (x^2, y) contains (x^4, y^2) we can leave out (x^2, y), giving an irreducible decomposition: $(x^4y, x^2y^2, y^3) = (x^2, y^3) \cap (x^4, y^2) \cap (y)$. The first two ideals are (x, y)-primary, so we can replace them by their intersection (x^4, x^2y^2, y^3). We can also replace them by (x^5, x^4y, x^2y^2, y^3) (for example) giving two minimal primary decompositions

$$(x^4y, x^2y^2, y^3) = (x^4, x^2y^2, y^3) \cap (y) = (x^5, x^4y, x^2y^2, y^3) \cap (y).$$

5. (a) Show that the radical of a primary ideal is prime.
(b) If I is a proper ideal containing a power $\mathfrak{m}^n$ of a maximal ideal $\mathfrak{m}$ show that I is primary and $\mathrm{rad}(I) = \mathfrak{m}$.
(c) Find an example of a power of a prime ideal that is not primary.

Solution. (a) The radical is the intersection of the minimal prime ideals containing the ideal and all of these prime ideals are associated primes, by a result in class. If the ideal is primary there is only one associated prime, so it must be the radical, which is prime.

(b) If I contains $\mathfrak{m}^n$ then $\mathfrak{m}$ is contained in the radical of I, and since it is maximal it must equal the radical of I. All associated primes of I contain the radical, so $\mathfrak{m}$ is the only one, and I is primary.
(c) Let $R = k[x, y]/(xy)$ and write the images of x and y in R as a and b. Then (a) and (a, b) are both prime ideals in R because $R/(a) \cong k[y]$ and $R/(a, b) \cong k$, which are integral domains. We see that $(a)^2$ is not primary because (a) is one of its associated primes, being minimal over its radical, but also (a, b) is an associated prime because it is the annihilator of $a + (a)^2$ in $R/(a)^2$. There are two associated primes so $(a)^2$ is not primary.

6. If P is a prime ideal of R the symbolic nth power of P is the ideal $P^{(n)} = P^n R_P \cap R$. Show that this is a primary ideal with radical P. (To assign meaning to the intersection, assume that R is a domain.)

Solution. Evidently $P^n \subseteq P^n R_P \cap R$ so P is contained in the radical of $P^{(n)}$, and it can be no larger because R/P is a domain. By Proposition 2.2(b) of Eisenbud the elements of $U = R - P$ act invertibly on $R/P^{(n)}$, so any annihilator of an element of $R/P^{(n)}$ is contained in P. Because associated primes contain the radical, P is the only associated prime and $P^{(n)}$ is primary.

Extra questions: do not upload to Gradescope.

7. (Eisenbud 3.7) Find an algorithm for computing the radical of a monomial ideal.

8. Let. I and J be ideals of a Noetherian ring R. Prove that if $JR_P \subset IR_P$ for every. $P \in \mathrm{Ass}_R(R/I)$ then $J \subset I$.

9. Let R be a Noetherian ring and let $x \in R$ be an element which is neither a unit nor a zero-divisor. Prove that $\mathrm{Ass}_R(R/xR) = \mathrm{Ass}_R(R/x^n R)$.