Math 8212 Commutative and Homological Algebra 2 Spring 2022

Homework Assignment 2 - Solutions Due Saturday 3/5/2022, uploaded to Gradescope.

Each question part is worth 1 point. There are 12 question parts. Assume that all categories are small. We define $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ to be the category whose objects are functors $\mathcal{C}\to\mathcal{D}$ and whose morphisms are natural transformations.

1. Suppose that $F:\mathcal{C}\to\mathcal{D}$ is an equivalence of categories.

(a) Show that, for all objects $x,y\in\mathrm{Ob}\mathcal{C}$, the functor F provides a bijection

$$\mathrm{Hom}_{\mathcal{C}}(x,y)\leftrightarrow \mathrm{Hom}_{\mathcal{D}}(F(x),F(y)),$$

that preserves composition, so that $\mathrm{End}_{\mathcal{C}}(x)\cong\mathrm{End}_{\mathcal{D}}(F(x))$ as monoids.

(b) Show that $x\cong y$ in $\mathcal{C}$ if and only if $F(x)\cong F(y)$ in $\mathcal{D}$, so that F provides a bijection between the isomorphism classes of $\mathcal{C}$, and of $\mathcal{D}$.

(c) Let $\mathcal{E}$ be a further category. Show that the functor categories $\mathrm{Fun}(\mathcal{C},\mathcal{E})$ and $\mathrm{Fun}(\mathcal{D},\mathcal{E})$ are naturally equivalent.

Solution. (a) For each morphism $\alpha:x\to y$ in $\mathcal{C}$ there is a morphism $F(\alpha):F(x)\to F(y)$, so F provides a mapping $\mathrm{Hom}_{\mathcal{C}}(x,y)\leftrightarrow\mathrm{Hom}_{\mathcal{D}}(F(x),F(y))$, and this mapping preserves composition. Because F is an equivalence, there is another functor $G:\mathcal{D}\to\mathcal{C}$ with natural isomorphisms $\theta:FG\to 1_{\mathcal{D}}$ and $\eta:GF\to 1_{\mathcal{C}}$. This means that for each $\alpha:x\to y$ in $\mathcal{C}$ we have $\alpha=\eta_y(GF(\alpha))\eta_x^{-1}$ which shows that $\alpha\mapsto F(\alpha)$ is one-to-one. Similarly the existence of θ shows that $\alpha\mapsto F(\alpha)$ is onto. Thus we have a bijection as claimed.
(b) If $x\cong y$ in $\mathcal{C}$ there are morphisms $\alpha:x\to y$ and $\beta:y\to x$ so that $\beta\alpha=1_x$ and $\alpha\beta=1_y$. Applying F these equations we get $F(\beta)F(\alpha)=1_{F(x)}$ and $F(\alpha)F(\beta)=1_{F(y)}$ so $F(x)\cong F(y)$. Conversely, using the functor G from part (a), if $F(x)\cong F(y)$ then $GF(x)\cong GF(y)$ by the argument already used. Now $\eta_z:GF(z)\to z$ is an isomorphism for all z, so $x\cong GF(x)\cong GF(y)\cong y$.
(c) With the previous notation, we get functors $F^*:\mathrm{Fun}(\mathcal{D},\mathcal{E})\to\mathrm{Fun}(\mathcal{C},\mathcal{E})$ and $G^*:\mathrm{Fun}(\mathcal{C},\mathcal{E})\to\mathrm{Fun}(\mathcal{D},\mathcal{E})$ given by precomposition with F and with G, so that if $\Phi:\mathcal{D}\to\mathcal{E}$ is a functor then $F^*(\Phi)=\Phi\circ F$ and similarly if $\Psi:\mathcal{C}\to\mathcal{E}$ is a functor then $G^*(\Psi)=\Psi\circ G$. We get a natural transformation $\eta^*:F^*G^*=(GF)^*\to 1_{\mathrm{Fun}(\mathcal{C},\mathcal{E})}$ given at a functor Ψ by $\eta^*_\Psi:(GF)^*(\Psi)=\Psi GF\to\Psi$ where $\eta^*_\Psi(x)=\Psi(\eta_x)$. This is a natural isomorphism with inverse given by a similar construction applied to the inverse of η. Also by a similar construction we get a natural isomorphism $\theta^*:G^*F^*=(FG)^*\to 1_{\mathrm{Fun}(\mathcal{D},\mathcal{E})}$ given at a functor $\Phi:\mathcal{D}\to\mathcal{E}$ by $\theta^*_\Phi(z)=\Phi(\theta_z)$. This shows that the functor categories are naturally equivalent.

2. Let $\mathcal{C}$ be a category and let $x,y\in\mathrm{Ob}\mathcal{C}$. Prove that if $x\cong y$ then $\mathrm{Hom}_{\mathcal{C}}(x,-)$ and $\mathrm{Hom}_{\mathcal{C}}(y,-)$ are naturally isomorphic functors $\mathcal{C}\to\mathrm{Set}$.

Solution. Let $\alpha : x \to y$ and $\beta : y \to x$ be inverse isomorphisms. For each object z in $\mathcal{C}$ we have mappings $\alpha_z^* : \mathrm{Hom}_{\mathcal{C}}(y,z) \to \mathrm{Hom}_{\mathcal{C}}(x,z)$ and $\beta_z^* : \mathrm{Hom}_{\mathcal{C}}(x,z) \to \mathrm{Hom}_{\mathcal{C}}(y,z)$ given by $\alpha_z^*(f) = f\alpha$ and $\beta_z^*(g) = g\beta$. Now α^* and β^* are natural transformations because if $u : z \to w$ then $u_*\alpha_z^*(f) = uf\alpha = \alpha_w^*(u_*(f))$, and similarly with β^*. They are inverse to each other because, for each z, we have $\beta_z^*\alpha_z^* = 1_{\mathrm{Hom}_{\mathcal{C}}(y,z)}$ and $\alpha_z^*\beta_z^* = 1_{\mathrm{Hom}_{\mathcal{C}}(x,z)}$.

3. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors and $\eta : F \to G$ a natural transformation.

(a) Show that if, for all $x \in \mathrm{Ob}\mathcal{C}$, the mapping $\eta_x : F(x) \to G(x)$ is an isomorphism in $\mathcal{D}$, then η is a natural isomorphism (meaning that it has a 2-sided inverse natural transformation $\theta : G \to F$).

(b) Suppose that F is an equivalence of categories and that F is naturally isomorphic to G, so $F \simeq G$. Show that G is an equivalence of categories.

Solution. (a) If η_x is an isomorphism for all x we may define mappings $\theta_x := \eta_x^{-1}$. These θ_x define a natural transformation $\theta : G \to F$ because we know that for all morphisms $\alpha : x \to y$ we have $G(\alpha)\eta_x = \eta_y F(\alpha)$, so that $\theta_y G(\alpha) = \eta_y^{-1} G(\alpha)\eta_x\eta_x^{-1} = \eta_y^{-1}\eta_y F(\alpha)\eta_x^{-1} = F(\alpha)\theta_x$. It is inverse to η.
(b) We will use the fact that if $F_1 : \mathcal{D} \to \mathcal{E}$ is another functor and $F \simeq G$ then $F_1F \simeq F_1G$. This is because the natural transformation $\eta : F \to G$ provides a natural transformation $F_1\eta : F_1F \to F_1G$ where, for each object x of $\mathcal{C}$, we have $(F_1\eta)_x = F_1(\eta_x) : F_1F(x) \to F_1G(x)$. If each η_x is an isomorphism, so is $(F_1\eta)_x$, because functors take isomorphisms to isomorphisms. We will take $F_1 : \mathcal{D} \to \mathcal{C}$ to be an inverse equivalence to F, so that $F_1F \simeq 1_{\mathcal{C}}$ and $FF_1 \simeq 1_{\mathcal{D}}$. Now $F_1G \simeq F_1F \simeq 1_{\mathcal{C}}$, and similarly $GF_1 \simeq FF_1 \simeq 1_{\mathcal{D}}$. Finally we show that equivalence $\simeq$ is transitive. Suppose we have natural equivalences $\eta : F \to G$ and $\theta : G \to H$. Then $\theta\eta : F \to H$ is a natural transformation, where $(\theta\eta)_x := \theta_x\eta_x$ and if both θ_x and η_x are isomorphisms, so is $(\theta\eta)_x$. It follows that $F_1G \simeq 1_{\mathcal{C}}$ and $GF_1 \simeq 1_{\mathcal{D}}$ so that F_1 is a natural inverse of G.

4. Let G be a group, which we regard as a category $\mathcal{G}$ with a single object, and with the elements of G as morphisms. Let $F : \mathcal{G} \to \mathcal{G}$ be a functor.

(a) Show that F is naturally isomorphic to the identity functor $1_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}$ if and only if the mapping $F : G \to G$, induced by F on the set of morphisms, is an inner automorphism; that is, an automorphism of the form $c_g : G \to G$ for some $g \in G$, where $c_g(h) = ghg^{-1}$ for all $h \in G$.

(b) Show that self equivalences of $\mathcal{G}$ are automorphisms of $\mathcal{G}$.

(c) Show that the group of natural isomorphism classes of self equivalences of $\mathcal{G}$ is isomorphic to $\mathrm{Aut}(G)/\mathrm{Inn}(G)$. (In the context of group theory, $\mathrm{Inn}(G)$ denotes the set of inner automorphisms of G, and $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the group of *outer* (or *non-inner*) automorphisms.)

Solution. (a) Writing the object of $\mathcal{G}$ as $*$, there is a natural isomorphism $\theta : F \simeq 1$ if and only if there is an isomorphism θ_* in $\mathcal{G}$ so that, for all morphisms α in $\mathcal{G}$ the following diagram commutes:

$$\begin{array}{ccc} * & \xrightarrow{F(\alpha)} & * \\ \downarrow{\scriptstyle \theta_*} & & \downarrow{\scriptstyle \theta_*} \\ * & \xrightarrow{\alpha} & * \end{array}$$

The morphisms θ_*, α and $F(\alpha)$ are all elements of G, and F is a homomorphism $G \to G$. The diagram means that F is c_g where $g = \theta_*$.
(b) If F is a self equivalence of $\mathcal{G}$ then $F(*) = *$ because $\mathcal{G}$ has only one object, and F is an isomorphism on the morphisms of $\mathcal{G}$ by exercise 1(a).
(c) Each automorphism F of $\mathcal{G}$ provides a bijective map f of the morphisms of $\mathcal{G}$ to itself preserving composition, so an automorphism $f : G \to G$, and from part (b) we see that the correspondence $F \leftrightarrow f$ is an isomorphism $\mathrm{Aut}(\mathcal{G}) \cong \mathrm{Aut}(G)$. We show that two automorphisms F_1, F_2 are equivalent if and only if the corresponding f_1, f_2 lie in the same coset of $\mathrm{Inn}(G)$. Now $F_1 \simeq F_2$ if and only if $F_2^{-1}F_1 \simeq 1_{\mathcal{G}}$ by an argument from question 3(b), which happens if and only if $f_2^{-1}f_1 \in \mathrm{Inn}(G)$ by part (a) or, in other words, f_1, f_2 lie in the same coset of $\mathrm{Inn}(G)$.

5. Let I be the poset with two elements 0 and 1, and with $0 < 1$. If P and Q are posets we can regard them as categories $\mathcal{P}$ and $\mathcal{Q}$ whose objects are the elements of the posets, and where there is a unique morphism $x \to y$ if and only if $x \le y$.

(a) Show that if P and Q are posets then a functor $\mathcal{P} \to \mathcal{Q}$ is 'the same thing as' an order-preserving map. (Don't worry about any fancy interpretation of 'the same thing as'!)

(b) Now consider two functors $F, G : \mathcal{P} \to \mathcal{Q}$, which we may regard as order-preserving maps $f, g : P \to Q$ by part (a). Show that the following three conditions are equivalent:
(i) there exists a natural transformation $F \to G$,
(ii) $f(x) \le g(x)$ for all $x \in P$,
(iii) there is an order-preserving map $h : P \times I \to Q$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in \mathcal{P}$. Here $P \times I$ denotes the product poset with order relation $(a_1, b_1) \le (a_2, b_2)$ if and only if $a_1 \le a_2$ and $b_1 \le b_2$, where $a_i \in P$ and $b_i \in I$.

Solution: (a) Let $F : \mathcal{P} \to \mathcal{Q}$ be a functor. If $x \le y$ in P we can regard this as a morphism $\alpha : x \to y$ in $\mathcal{P}$, so that $F(\alpha) : F(x) \to F(y)$ is a morphism in $\mathcal{Q}$, and $F(x) \le F(y)$. Thus F is an order preserving map. Conversely, given an order preserving map $f : P \to Q$ we obtain a functor $\mathcal{P} \to \mathcal{Q}$ that on objects is the same as f, and where if $x \to y$ is a morphism in $\mathcal{P}$ we define the effect of the functor to be the unique morphism $f(x) \to f(y)$, which exists because $f(x) \le f(y)$.
(b) Suppose (i) Then for each object x of $\mathcal{P}$ there is a morphism $\tau_x : F(x) \to G(x)$, so that $f(x) \le g(x)$ for all $x \in P$. Thus (ii) holds.

Assuming (ii) holds, we show that the mapping defined in (iii) is order preserving. Suppose that $(a_1, b_1) \le (a_2, b_2)$ and apply h. If $b_1 = b_2$ then $h(a_1, b_1) \le h(a_2, b_2)$ because either f or g is order preserving. The other possibility is $b_1 = 0$ and $b_2 = 1$, in which case this inequality holds because, in addition, $f(x) \le g(x)$ for all x. Thus (iii) holds.
Assuming (iii) we define τ_x to be the unique map $f(x) \to g(x)$, which exists because $f(x) = h(x, 0) \le h(x, 1) = g(x)$. This shows that (i) holds.

6. Let $1_{R-\mathrm{mod}} : R\text{-mod} \to R\text{-mod}$ denote the identity functor. Let $\mathrm{Nat}(1_{R-\mathrm{mod}}, 1_{R-\mathrm{mod}})$ denote the set of natural transformations from this functor to itself, noting that this set has the structure of a ring (multiplication is composition and addition comes because we can add homomorphisms of R-modules, so that for two natural transformations θ, ψ at an object x we have $(\theta + \psi)_x = \theta_x + \psi_x$). Show that $\mathrm{Nat}(1_{R-\mathrm{mod}}, 1_{R-\mathrm{mod}}) \cong Z(R)$.

Solution. We define mappings

$$f : \mathrm{Nat}(1_{R-\mathrm{mod}}, 1_{R-\mathrm{mod}}) \to R \quad \text{and} \quad g : Z(R) \to \mathrm{Nat}(1_{R-\mathrm{mod}}, 1_{R-\mathrm{mod}})$$

as follows. If η is such a natural transformation note that $\eta_R : R \to R$ is an R-module homomorphism. We put $f(\eta) = \eta_R(1_R) \in R$. If $r \in Z(R)$ we define $g(r)$ to be the natural transformation with $g(r)_M : M \to M$ the mapping $m \mapsto rm$. This is an R-module homomorphism because r lies in the center of R, and $g(r)$ is a natural transformation because if $\alpha : M \to M$ is a homomorphism of R-modules then $g(r)_M \alpha(m) = r\alpha(m) = \alpha(rm) = \alpha g(r)_M(m)$. We should verify several more things: $f(\eta)$ lies in $Z(R)$ and the two composite mappings fg and gf are the identity. If x is any element of R we have an R-module homomorphism $\mu_x : R \to R$ where $\mu_x(s) = sx$. Naturality of η means that $\eta_R \mu_x = \mu_x \eta_R$. Applying these to $1 \in R$ we get $\eta_R(x) = \eta_R(x1) = x\eta_R(1) = \eta_R(1)x$, showing $f(\eta)$ lies in $Z(R)$. It is immediate that fg is the identity on $Z(R)$. Finally we show that $\eta = g(\eta_R(1_R))$ to see this consider the commutative diagram of R-modules

$$\begin{array}{ccc} R & \xrightarrow{\eta_R} & R \\ \downarrow & & \downarrow \\ M & \xrightarrow{\eta_M} & M \end{array}$$

where the vertical arrows are determined by $1 \mapsto m$ for some arbitrary element $m \in M$. Commutativity shows that $\eta_M(m) = \eta_R(1_R)m$, which is what is needed.

Extra question: do not upload to Gradescope.

7. Let $\mathcal{C}$ be a small category and let $F, G : \mathcal{C} \to \mathrm{Set}$ be functors. Show that a natural transformation of functors $\tau : F \to G$ is an epimorphism in $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ if and only if for every object x of $\mathcal{C}$, $\tau_x : F(x) \to G(x)$ is a surjection; and it is a monomorphism if and only if for every object x of $\mathcal{C}$, $\tau_x : F(x) \to G(x)$ is a 1-1 map.

8. Write out a proof that if G is the right adjoint of a functor F with the property that F preserves monomorphisms, then G sends injective objects to injective objects.

9. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors with F left adjoint to G, and with adjunction unit η and counit ϵ. Write out a proof that the second triangular identity holds, namely the following triangle commutes:

$$\begin{array}{ccccc} G & & \xrightarrow{1_G} & & G \\ & \searrow_{\eta_G} & & \nearrow_{G\epsilon} & \\ & & GFG & & \end{array}$$

10. Assume the axiom of choice in this question, or else make some assumption such as: everything is finite. Let $\mathcal{C}$ be a category, and for each isomorphism class $\hat{x}$ of objects x, choose a fixed representative $u_{\hat{x}}$. For each object x choose a fixed isomorphism $i_x : x \to u_{\hat{x}}$. Let $\mathcal{D}$ be the full subcategory whose objects are the $u_{\hat{x}}$ where $x \in \mathrm{Ob}\mathcal{C}$. 'Full' means that for each pair of objects y, z of $\mathcal{D}$ we have $\mathrm{Hom}_{\mathcal{D}}(y, z) = \mathrm{Hom}_{\mathcal{C}}(y, z)$. Define $F(x) = \hat{x}$, and for each morphism $\alpha : x \to y$ define $F(\alpha) : F(x) \to F(y)$ to be $i_y \alpha i_x^{-1}$.

(a) Show that F is a functor.

(b) Show that F and the inclusion functor $\mathrm{inc} : \mathcal{D} \to \mathcal{C}$ are inverse equivalences of categories $\mathcal{D} \simeq \mathcal{C}$. (It will help to assume that when $x = u_{\hat{x}}$, the chosen isomorphism is the identity 1_x.)

(c) Deduce that the category Set of finite sets is equivalent to the category with objects $\mathbb{N} := \{0, 1, 2, \ldots\}$ and where $\mathrm{Hom}(n, m)$ is the set of all mappings of sets from $\mathbf{n} := \{1, \ldots, n\}$ to $\mathbf{m} := \{1, \ldots, m\}$. We take $\mathbf{0} = \emptyset$.

(d) Deduce also the following: let K be a field. Show that the category Vec of finite dimensional vectors spaces over K is equivalent to the category $\mathcal{C}$ with objects $\mathbb{N} := \{0, 1, 2, \ldots\}$, where $\mathrm{Hom}_{\mathcal{C}}(n, m)$ is the set $M_{m,n}(K)$ of $m \times n$ matrices with entries in K, and where composition of morphisms is matrix multiplication. In case m or n is zero, give a definition of $\mathrm{Hom}_{\mathcal{C}}(n, m)$ that will make this question make sense.

11. Let $\mathcal{C}$ be a small category. A *self-equivalence* of $\mathcal{C}$ is an equivalence of categories $F : \mathcal{C} \to \mathcal{C}$. Show that the set of natural isomorphism classes of self equivalences of $\mathcal{C}$ is a group, with multiplication induced by composition of functors.