On the definition of the limit, there
we have 
\[ \lim_{x \to a} f(x) = L \] 
if and only if for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that
\[ 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon \]

Prove: let \( \varepsilon > 0 \). Take \( \delta = \frac{\varepsilon}{2} \).
Then for \( 0 < |x - a| < \delta \),
we have
\[ |f(x) - L| = |(f(x) - L) - (f(a) - L)| = \frac{|f(x) - f(a)|}{2} \]
\[ < \frac{\varepsilon}{2} \cdot \frac{1}{2} = \varepsilon \]

Show: \( \lim_{x \to a} f(x) = 0 \)

Theorem 1: \( \lim_{x \to a} f(x) = \frac{1}{x} \) and \( \lim_{x \to a} g(x) = 1 \)
then \( \lim_{x \to a} (f(x) + g(x)) = \frac{1}{x} + 1 \)

Proof: \( |f(x) - \frac{1}{x}| = \frac{|x - a|}{x^2} \leq \frac{|x - a|}{a^2} \)
Since for \( x \neq a \), \( |x - a| < a \)
\[ \Rightarrow |f(x) - \frac{1}{x}| < \frac{1}{a^2} |x - a| \]
\[ < \varepsilon \]
for \( \frac{1}{a^2} > \varepsilon \)

(\( \delta \) is not to be shown)

Remark: \( \lim_{x \to a} \frac{1}{x} \) is discontinuous, but \( \lim_{x \to a} |x| = a \)

4.1 - 4.2_10

Monday, October 14, 2019
10:18 PM