Mean Value Theorem: Let $f\in C[a, b]$ be a continuous function. Then there is a point $c\in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

**Proof:**

It is enough to show that if $f(x) = f(0)$, then $y = 0$ is the line $f(x) = 0$.

**Why:**

The key step is to show that the function $y - f(0) = f(x) - f(0)$ is continuous on $[a, b]$. If it is not, then some part of $[a, b]$ is not in $[a, b]$.

To show this, use the fact that a continuous function on a compact set has a maximum or a minimum.

If one of these is not an endpoint, then $f$ has a derivative $f'(c)$ for some $c \in [a, b]$.

*If both are endpoints, $f$ is constant and we have been derivatives.*

**Taylor's Theorem:** Let $f$ be in $C[k, c]$, a derivation in $C[a, b]$ and differentiable on $(a, b)$, let $x \in (a, b)$. Then for each $x \in (a, b)$, with $x \neq c$, there is a $k$ between $x$ and $c$ such that:

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(k)}{2}(x - c)^2 + \ldots + \frac{f^{(n)}(k)}{n!}(x - c)^n$$

**Example:**

Let $f(x) = e^x$ on $[0, 1]$, $f(x) = e^x$, $f(x) = e^x$, $f(x) = e^x$.

Take $x = 0$.

$$f(x) = f(0) + f'(0)(x - 0) + \frac{f''(k)}{2}(x - 0)^2 + \ldots$$

Taylor polynomial remainder

**Remark:** This is how your calculator produces long decimal expansions for things like $e^x$ and sin(x).

**Example:**

Find the $4$th Taylor polynomial at $x = 0$.

$$f(0) = 1$$

$$f'(0) = 1$$

$$f''(0) = 2$$

$$f'''(0) = 6$$

$$f^{(4)}(0) = 24$$

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

Since $x \in [-1, 1]$ then remainder is $\frac{x^5}{120} < 0.0005$.

So for $x \in (-1, 1)$, $1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ approximates $e^x$ within 0.0005.