Chapter 3 Review

For each of the following statements, determine whether it is true or false. If the statement is true, give a proof. If the statement is false, give a counterexample. You may use any result proved in class or in the textbook.

(a) If $S$ is a subset of $\mathbb{N}$ then $S$ has a maximum.

(b) For all $n \in \mathbb{N}$, $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n + 1)^2$.

(c) If $x$ and $y$ are real numbers with $x < y$ then there is a rational number $q$ and an irrational number $r$ with $x < q < r < y$.

(d) If $S \subseteq \mathbb{R}$ is unbounded, then $\mathbb{R} \setminus S$ is bounded.

(e) Let $S \subseteq \mathbb{R}$ be a nonempty bounded set and let $m = \sup S$. Then $m$ is a boundary point of $S$.

(f) For any subsets $A$, $B$ of $\mathbb{R}$, $\overline{A \cap B} = \overline{A} \cap \overline{B}$. Here, $\overline{A}$ denotes the closure of the set $A$.

(g) Let $\{A_i : i \in I\}$ be a (not necessarily countable) family of open sets. Let $A = \bigcup_{i \in I} A_i$. Then the set $B = [0, 1] \cap (\mathbb{R} \setminus A)$ is compact.

(h) Any set that has a minimum and a maximum is compact.

Complete each of the following definitions.

(a) We say $S \subseteq \mathbb{R}$ is bounded if . . .

(b) Let $S \subseteq \mathbb{R}$ be bounded below. We say $m$ is the infimum of $S$ if . . .

(c) The deleted neighborhood of $x$ of radius $\varepsilon$ is the set . . .

(d) A point $x$ is an accumulation point of $S$ if . . .

(e) We say $\mathcal{G}$ is an open cover of $S$ if . . .

(f) A set $S \subseteq \mathbb{R}$ is compact if . . .
Solutions

(a) If $S$ is a subset of $\mathbb{N}$ then $S$ has a maximum.

False.

Let $S$ be the set of even natural numbers. This set is not bounded above, so it does not have a maximum.

(b) For all $n \in \mathbb{N}$, $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$.

True.

We prove this by induction.

Base case: When $n = 1$, the left side is $1^3$ and the right side is $\frac{1}{4}1^2(1+1)^2 = \frac{1}{4} \cdot 4 = 1$. So the statement is true at $n = 1$.

Inductive step: Let $n \in \mathbb{N}$ and suppose that

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$$

We add $(n+1)^3$ to both sides and simplify:

$$\sum_{i=1}^{n+1} i^3 = \frac{1}{4}n^2(n+1)^2 + (n+1)^3$$

$$= \frac{n^2(n+1)^2 + 4(n+1)^3}{4}$$

$$= \frac{(n+1)^2(n^2 + 4(n + 1))}{4}$$

$$= \frac{(n+1)^2(n+2)^2}{4}$$

Thus the statement holds for $n + 1$, so by induction it holds for all $n \in \mathbb{N}$.

(c) If $x$ and $y$ are real numbers with $x < y$ then there is a rational number $q$ and an irrational number $r$ with $x < q < r < y$.

True.

By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational number $q$ with $x < q < y$. But then by the density of the irrationals in $\mathbb{R}$, there is an irrational $r$ with $q < r < y$. So we have $x < q < r < y$ as needed.

(d) If $S \subseteq \mathbb{R}$ is unbounded, then $\mathbb{R} \setminus S$ is bounded.

False.

Consider the set $S = \mathbb{Z}$. This is unbounded, but its complement $\mathbb{R} \setminus \mathbb{Z}$ contains \( \{ n + \frac{1}{2} : n \in \mathbb{Z} \} \) which is an unbounded set.
(e) Let $S \subseteq \mathbb{R}$ be a nonempty bounded set and let $m = \sup S$. Then $m$ is a boundary point of $S$.

True.

Let $m = \sup S$. We need to show that for every $\varepsilon > 0$, $N(m, \varepsilon) \cap S \neq \emptyset$ and $N(m, \varepsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset$.

Given any $\varepsilon > 0$, since $m$ is $\sup S$ there is $x \in S$ so that $m - \varepsilon < x \leq m$. But then $x \in N(m, \varepsilon)$ so $N(m, \varepsilon) \cap S \neq \emptyset$.

Let $y = m + \frac{\varepsilon}{2}$. Since $y > m$ and $m$ is an upper bound for $S$, $y \notin S$. But $m \leq y < m + \varepsilon$ so $y \in N(m, \varepsilon)$ and thus $N(m, \varepsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset$.

So $m$ is a boundary point of $S$.

(f) For any subsets $A, B$ of $\mathbb{R}$, $\overline{A} \cap \overline{B} = \overline{A \cap B}$. Here, $\overline{A}$ denotes the closure of the set $A$.

False.

Let $A = (0, 1)$ and $B = (1, 2)$. We have $A \cap B = \emptyset$ so since the empty set is closed, $\overline{A} \cap \overline{B} = \emptyset$. On the other hand, we know from examples in class that $\overline{A} = [0, 1]$ and $\overline{B} = [1, 2]$ so $\overline{A} \cap \overline{B} = \{1\}$.

(g) Let $\{A_i : i \in I\}$ be a (not necessarily countable) family of open sets. Let $A = \bigcup_{i \in I} A_i$.

Then the set $B = [0, 1] \cap (\mathbb{R} \setminus A)$ is compact.

True.

Since each $A_i$ is open, the union $A$ is open as well. Thus $\mathbb{R} \setminus A$ is closed by definition. Since $[0, 1]$ is closed, $B = [0, 1] \cap (\mathbb{R} \setminus A)$ is closed. Now $B \subseteq [0, 1]$ and $[0, 1]$ is bounded, so $B$ is also bounded. Thus $B$ is compact by the Heine-Borel theorem.

(h) Any set that has a minimum and a maximum is compact.

False.

Consider the set $S = 0, 3 \cup (1, 2)$. This set clearly has minimum 0 and maximum 3. However, it is not closed since 2 is a boundary point of $S$ but 2 $\notin S$. So by the Heine-Borel theorem, $S$ is not compact.

(a) We say $S \subseteq \mathbb{R}$ is bounded if there are $M, N \in \mathbb{R}$ so that for all $s \in S$, $s \leq M$ and $s \geq N$.

(b) Let $S \subseteq \mathbb{R}$ be bounded below. We say $m$ is the infimum of $S$ if for all $s \in S$, $m \leq s$ and whenever $m' > m$ there is $s' \in S$ so that $m' < s'$.

(c) The deleted neighborhood of $x$ of radius $\varepsilon$ is the set $\{y \in \mathbb{R} : 0 < |x - y| < \varepsilon\} = \{y \in \mathbb{R} : x - \varepsilon < y < x \text{ or } x < y < x + \varepsilon\}$.
(d) A point $x$ is an accumulation point of $S$ if for every $\varepsilon > 0$, $N^*(x, \varepsilon) \cap S \neq \emptyset$.

(e) We say $\mathcal{G}$ is an open cover of $S$ if $S \subseteq \bigcup_{A \in \mathcal{G}} A$ and each $A \in \mathcal{G}$ is an open set.

(f) A set $S \subseteq \mathbb{R}$ is compact if every open cover of $S$ has a finite subcover.