Determinants_1-21

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Determinants

Given a square matrix A, we associate a number $\det(A)$

That is, we will have a function
$\det: \mathrm{Mat}(n \times n) \longrightarrow \mathbb{R}$

We will take an axiomatic approach to the determinant, demanding that it satisfy certain properties and looking at the consequences before giving an actual formula (or even establishing that such a function exists)

Notation: Given $\vec{a}_1, \dots, \vec{a}_n \in \mathbb{R}^n$, write $(\vec{a}_1, \dots, \vec{a}_n)$ for the matrix with i^{th} row $\vec{a}_i$.

Def: We say $d: \mathrm{Mat}(n \times n) \to \mathbb{R}$ is a <u>determinant</u> if it satisfies each of the following:

1. For all $\lambda \in \mathbb{R}$
$d(\vec{a}_1, \dots, \lambda \vec{a}_i, \dots, \vec{a}_n) = \lambda \cdot d(\vec{a}_1, \dots, \vec{a}_n)$
2. $d(\vec{a}_1, \dots, \vec{a}_i + \vec{b}, \dots, \vec{a}_n) = d(\vec{a}_1, \dots, \vec{a}_n) + d(\vec{a}_1, \dots, \vec{b}, \dots, \vec{a}_n)$

→ d is linear in each row
"d is a multilinear function in rows"

3. If two rows of A are equal, then $d(A) = 0$
If $\vec{a}_i = \vec{a}_j$ for some $i \neq j$, $d(\vec{a}_1, \dots, \vec{a}_n) = 0$.
4. $d(I_n) = 1$
$d(\vec{e}_1, \dots, \vec{e}_n) = 1$

<u>Theorem</u>: If $d: \mathrm{Mat}(n \times n) \to \mathbb{R}$ is a determinant function then

1. If $\vec{a}_i = \vec{0}$, $d(\vec{a}_1, \dots, \vec{a}_n) = 0$.
2. Switching rows changes sign, i.e. for $i \neq j$
$d(\vec{a}_1, \dots, \vec{a}_i, \dots, \vec{a}_j, \dots, \vec{a}_n) = -d(\vec{a}_1, \dots, \vec{a}_j, \dots, \vec{a}_i, \dots, \vec{a}_n)$
3. If $\vec{a}_1, \dots, \vec{a}_n$ are linearly dependent then $d(\vec{a}_1, \dots, \vec{a}_n) = 0$.

Proof:

(1) $d(\vec{a}_1, \dots, \vec{0}, \dots, \vec{a}_n) = d(\vec{a}_1, \dots, 0 \cdot \vec{0}, \dots, \vec{a}_n)$
$= 0 \cdot d(\vec{a}_1, \dots, \vec{a}_n) = 0$.

(2) $0 = d(\vec{a}_1 + \vec{a}_2, \vec{a}_1 + \vec{a}_2)$
$= d(\vec{a}_1, \vec{a}_1 + \vec{a}_2) + d(\vec{a}_2, \vec{a}_1 + \vec{a}_2)$
$= d(\vec{a}_1, \vec{a}_1) + d(\vec{a}_1, \vec{a}_2) + d(\vec{a}_2, \vec{a}_1) + d(\vec{a}_2, \vec{a}_2)$
$= d(\vec{a}_1, \vec{a}_2) + d(\vec{a}_2, \vec{a}_1)$

(3) Suppose $\vec{a}_1 = \sum_{i=2}^{n} c_i \vec{a}_i$

$d\left(\sum_{i=2}^{n} c_i \vec{a}_i, \vec{a}_2, \dots, \vec{a}_n\right)$
$= \sum_{i=2}^{n} c_i \, d(\vec{a}_i, \vec{a}_2, \dots, \vec{a}_n)$
$= \sum_{i=2}^{n} c_i \cdot 0 = 0$

Using what we know so far, we can compute some determinants already. If a determinant function exists, it will have to look a certain way.

Ex: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det(a\vec{i} + b\vec{j}, c\vec{i} + d\vec{j})$
$= ac \det(\vec{i}, \vec{i}) + ad \det(\vec{i}, \vec{j}) + bc \det(\vec{j}, \vec{i}) + bd \det(\vec{j}, \vec{j})$
$= ad - bc$

Remark: For $n = 2$, if we define
$\det: \mathrm{Mat}(2 \times 2) \to \mathbb{R}$
by $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$, it satisfies all the necessary properties of a determinant.

Ex: If $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$ is a diagonal matrix

then $\det A = \prod_{i=1}^{n} \lambda_i = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$

$\det(A) = \det(\lambda_1 \vec{e}_1, \dots, \lambda_n \vec{e}_n)$
$= \left[\prod_{i=1}^{n} \lambda_i\right] \det(\vec{e}_1, \dots, \vec{e}_n)$
$= \prod_{i=1}^{n} \lambda_i$

Recall: Gauss-Jordan elimination

→ Switch rows
Scale det by (-1)

→ Scale a row by $\lambda \neq 0$
Scale det by $1/\lambda$
$\det(\vec{a}_1, \dots, \lambda \vec{a}_i, \dots, \vec{a}_n) = \lambda \det(\vec{a}_1, \dots, \vec{a}_n)$

→ Add a multiple of one row to another
NO CHANGE!
$\det(\vec{a}_1, \dots, \vec{a}_i + \lambda \vec{a}_j, \dots, \vec{a}_j, \dots, \vec{a}_n)$
$= \det(\vec{a}_1, \dots, \vec{a}_n) + \lambda \det(\vec{a}_1, \dots, \vec{a}_j, \dots, \vec{a}_j, \dots, \vec{a}_n)$
$= \det(\vec{a}_1, \dots, \vec{a}_n)$

Ex: If $A = \begin{pmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$ is upper triangular,

then $\det A = \prod_{i=1}^{n} \lambda_i$

Ex: $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix}$ $\quad -4(\text{row } 1) + (\text{row } 2)$, $-7(\text{row } 1) + (\text{row } 3)$

$= \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{vmatrix}$ $\quad -2(\text{row } 2) + (\text{row } 3)$

$= 0$

Ex: $\begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 1 & 1 \end{vmatrix}$ $\quad -2(\text{row } 1) + (\text{row } 2)$

$= \begin{vmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{vmatrix}$ $\quad (\text{row } 2) + (\text{row } 3)$

$= -3$

We know that every matrix can be reduced to an upper triangular matrix using Gaussian elimination. So from what we have showed so far, if a determinant exists it is unique.

Next goal: $\det(AB) = \det(A) \cdot \det(B)$