InnerProd_2-4

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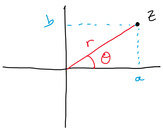
Complex Numbers

$z = a + bi = r\cos\theta + ir\sin\theta$



The modulus or absolute value of z is
$r = |a+bi| = \sqrt{a^2+b^2}$

An argument for $a+bi$ satisfies
$a = r\cos\theta$ $b = r\sin\theta$, and
often also $-\pi < \theta \le \pi$.

The complex conjugate of $z = a+bi$
is $\bar{z} = a - bi$

Properties:

For $z \neq 0$, $1/z = \bar{z}/|z|$ $|z|^2 = z\bar{z}$

$|z_1 + z_2| \le |z_1| + |z_2|$ $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$|z_1 z_2| = |z_1| \cdot |z_2|$ $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

$|z_1/z_2| = |z_1|/|z_2|$ $\overline{z_1/z_2} = \bar{z}_1/\bar{z}_2$

If f is a polynomial with real coefficients and root $z \in \mathbb{C}$, $\bar{z}$ is also a root of f.

For $z = a+bi$, we define

$e^z = e^a(\cos b + i\sin b)$

Properties:

$e^{z_1} e^{z_2} = e^{z_1 + z_2}$

Every $z \in \mathbb{C}$ can be expressed as $z = re^{i\theta}$
where $r = |z|$ and $\theta = \arg(z)$

$re^{i\theta} \cdot se^{i\varphi} = (rs)e^{i(\theta+\varphi)}$

$1/re^{i\theta} = \frac{1}{r}e^{-i\theta}$

$(re^{i\theta})^n = r^n e^{in\theta}$

An inner product $\langle \cdot, \cdot \rangle$ on a complex vector space V:

0. $\langle x, y \rangle \in \mathbb{C}$
1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ conjugate symmetry
2. $\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$ linear in first entry
   $\langle c \cdot x, y \rangle = c \cdot \langle x, y \rangle$
3. $\langle x, x \rangle > 0$ for $x \neq 0$ positivity

Remark: $\langle x, cy \rangle = \bar{c}\langle x, y \rangle$

Ex: On $\mathbb{C}^3$

$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1\bar{y}_1 + x_2\bar{y}_2 + x_3\bar{y}_3$

This is the complex version of the dot product and it makes $\mathbb{C}^3$ a Euclidean space. If we restrict to $\mathbb{R}^3 \subset \mathbb{C}^3$, we can recover results about real vector spaces.

Theorem: Let $(V, \langle , \rangle)$ be a Euclidean space and $T: V \to V$ a linear map. Let λ be an eigenvalue corresponding to $x \in V$. Then $\lambda = \frac{\langle T(x), x \rangle}{\langle x, x \rangle}$.

Def: Let V be Euclidean and $T: V \to V$ linear.
T is Hermitian if $\langle T(x), y \rangle = \langle x, T(y) \rangle$
T is skew-Hermitian if $\langle T(x), y \rangle = -\langle x, T(y) \rangle$

If V is real, we call these symmetric and skew-symmetric

Theorem: Let V be a real Euclidean space and $T: V \to V$ skew-symmetric. If $\lambda \in \mathbb{R}$ is an eigenvalue then $\lambda = 0$

Theorem: Let V be a (complex) Euclidean space and $T: V \to V$ linear. Suppose λ is an eigenvalue of T.
If T is Hermitian, $\lambda = \bar{\lambda}$ so $\lambda \in \mathbb{R}$
If T is skew-Hermitian, $\lambda = -\bar{\lambda}$ so $\lambda \in i\mathbb{R}$

Proof: We know $\lambda = \frac{\langle T(x), x \rangle}{\langle x, x \rangle}$ so $\bar{\lambda} = \frac{\langle x, T(x) \rangle}{\langle x, x \rangle}$

If T is skew-symmetric, $\frac{\langle T(x), x \rangle}{\langle x, x \rangle} = -\frac{\langle x, T(x) \rangle}{\langle x, x \rangle}$
so $\lambda = -\bar{\lambda}$. Since $\lambda \in \mathbb{R}$, $-\bar{\lambda} = -\lambda$.
So $\lambda = -\lambda$ and $\lambda = 0$.

If T is Hermitian, $\frac{\langle T(x), x \rangle}{\langle x, x \rangle} = \frac{\langle x, T(x) \rangle}{\langle x, x \rangle}$
so $\lambda = \bar{\lambda}$

If T is skew-Hermitian, $\frac{\langle T(x), x \rangle}{\langle x, x \rangle} = -\frac{\langle x, T(x) \rangle}{\langle x, x \rangle}$
so $\lambda = -\bar{\lambda}$