Limits_2-14

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Chapter 8: Differential Calculus of Scalar + Vector Fields

We deal with (nonlinear) functions

$f: \mathbb{R}^n \to \mathbb{R}^m$

$f: U \to \mathbb{R}^m$ where $U \subseteq \mathbb{R}^n$

$f: \mathbb{R}^n \to \mathbb{R}$ is called a <u>scalar field</u>

$f: \mathbb{R}^n \to \mathbb{R}^m$ $(m > 1)$ is called a <u>vector field</u>

Notation: $f(\vec{x}) = f(x) = f(x_1, \dots, x_n)$

Dot product $\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y}$

Norm $\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = |\vec{x}|$

The <u>open ball</u> centered at $\vec{a} \in \mathbb{R}^n$ of radius r is

$B_r(\vec{a}) = \{ \vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{a}\| < r \}$

p is called an <u>interior point</u> of $U \subset \mathbb{R}^n$ if

$\exists r > 0$ so that $B_r(p) \subset U$

U is <u>open</u> if every point in U is interior.

Ex: $[0,1] \times [0,1] \subseteq \mathbb{R}^2$ is <u>not</u> open

$(0,1) \times (0,1) \subseteq \mathbb{R}^2$ is open

If $A, B \subseteq \mathbb{R}$ are open, $A \times B \subseteq \mathbb{R}^2$ is open.

A point $\vec{x} \in \mathbb{R}^n$ is a <u>boundary point</u> of U if $\forall r > 0$ we have $B_r(\vec{x}) \cap U \neq \emptyset$ and $B_r(\vec{x}) \cap U^c \neq \emptyset$

Let ∂U be the <u>boundary</u> of U

A set U is <u>closed</u> if U^c is open

This occurs iff $\partial U \subset U$

Def: Given $f: U \to \mathbb{R}^m$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$

we say $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = \vec{b}$ if

$\forall \varepsilon > 0$ $\exists r > 0$ s.t. if $\vec{x} \in B_r(\vec{a})$ then $f(\vec{x}) \in B_\varepsilon(\vec{b})$

if $\|\vec{x} - \vec{a}\| < r$ then $\|f(\vec{x}) - \vec{b}\| < \varepsilon$

We say f is <u>continuous</u> at $\vec{x}_0 \in U$ if

$\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$

<u>Limit Laws</u>

Assume $\lim_{x \to a} f(x) = b$ $\lim_{x \to a} g(x) = c$

1. $\lim_{x \to a} (f(x) + g(x)) = b + c$
2. For $\lambda \in \mathbb{R}$, $\lim_{x \to a} \lambda \cdot f(x) = \lambda b$
3. $\lim_{x \to a} (f(x) \cdot g(x)) = b \cdot c$
4. $\lim_{x \to a} \|f(x)\| = \|b\|$

<u>Lemma</u>: Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be $f(\vec{x}) = (f_1(\vec{x}), \dots, f_m(\vec{x}))$.

f is continuous at $\vec{x}_0$ iff each $f_i: \mathbb{R}^n \to \mathbb{R}$ is cont. at $\vec{x}_0$

Proof: Note: $f_i(\vec{x}) = f(\vec{x}) \cdot \vec{e}_i$ and $f(\vec{x}) = \sum_{i=1}^{m} f_i(\vec{x}) \vec{e}_i$

Assume f is continuous at $\vec{x}_0$. Then

$$\lim_{\vec{x} \to \vec{x}_0} f_i(\vec{x}) = \left[\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) \right] \cdot \left[\lim_{\vec{x} \to \vec{x}_0} \vec{e}_i \right]$$
$$= f(x_0) \cdot \vec{e}_i$$
$$= f_i(\vec{x}_0)$$

Assume each f_i is continuous at $\vec{x}_0$. Then

$$\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = \sum_{i=1}^{n} \left[\lim_{\vec{x} \to \vec{x}_0} f_i(\vec{x}) \right] \cdot \vec{e}_i$$
$$= \sum_{i=1}^{n} f_i(\vec{x}_0) \vec{e}_i$$
$$= f(\vec{x}_0)$$

Ex: All linear functions $T: \mathbb{R}^n \to \mathbb{R}^m$ are continuous on all of $\mathbb{R}^n$

All polynomial functions $f: \mathbb{R}^n \to \mathbb{R}$ are continuous on all of $\mathbb{R}^n$

If p, q are polynomials in $\mathbb{R}^n$, $\frac{p(\vec{x})}{q(\vec{x})}$ is continuous wherever $q(\vec{x}) \neq 0$.

The projections $p_i: \mathbb{R}^n \to \mathbb{R}$ $p_i(\vec{x}) = x_i$ and inclusions $\iota_j: \mathbb{R} \to \mathbb{R}^m$ $\iota_j(x) = x \vec{e}_j$ are continuous.

<u>Proposition</u>: If $f: \mathbb{R}^n \to \mathbb{R}^m$ and $g: \mathbb{R}^m \to \mathbb{R}^\ell$ are both continuous, $g \circ f: \mathbb{R}^n \to \mathbb{R}^\ell$ is continuous.

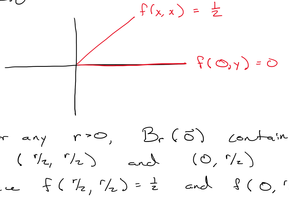
Ex: $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq \vec{0} \\ 0 & (x,y) = \vec{0} \end{cases}$

For all $(x_0, y_0) \in \mathbb{R}^2$

$\lim_{x \to x_0} f(x, y_0) = f(x_0, y_0)$ and $\lim_{y \to y_0} f(x_0, y) = f(x_0, y_0)$

But!

$\lim_{(x,y) \to \vec{0}} f(x,y)$ does not exist



For any $r > 0$, $B_r(\vec{0})$ contains $(1/2, 1/2)$ and $(0, 1/2)$

Since $f(1/2, 1/2) = \frac{1}{2}$ and $f(0, 1/2) = 0$, these will not both be in any $B_\varepsilon(c)$ for $\varepsilon < 1/4$.

Derivative of a Scalar Field w/r/t a vector

$U \subset \mathbb{R}^n$ $f: U \to \mathbb{R}$

Fix $\vec{a} \in \operatorname{Int} U$. We want to understand how f changes as we move from $\vec{a}$ in a certain direction. So fix $\vec{y} \in \mathbb{R}^n$. For $h \in \mathbb{R}$ small enough, $\vec{a} + h\vec{y} \in U$.

$$D_{\vec{y}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{y}) - f(\vec{a})}{h}$$