4.4: 1, 4, 5, 7, 11
Solutions for 4.4 courtesy of Jorin Schug

1. (a) If T has an eigenvalue λ, prove that aT has the eigenvalue $a\lambda$.

   Let x be an eigenvector of T corresponding to λ. We show that $(aT)(x) = (a\lambda)x$. Indeed
   $$(aT)(x) = aT(x) = a\lambda x = (a\lambda)x.$$
   This proves the claim.

   (b) If x is an eigenvector for both T_1 and T_2, prove that x is also an eigenvector for $aT_1 + bT_2$. How are the eigenvalues related?

   Let x be an eigenvector corresponding to λ_1 and λ_2 (for T_1 and T_2 respectively). We claim that x is an eigenvector corresponding to $\lambda = a\lambda_1 + b\lambda_2$. Indeed
   $$(aT_1 + bT_2)(x) = aT_1(x) + bT_2(x) = a\lambda_1 x + b\lambda_2 x = (a\lambda_1 + b\lambda_2)x.$$

4. If $T\colon V \to V$ has the property that T^2 has a nonnegative eigenvalue λ^2, prove that at least one of λ or $-\lambda$ is an eigenvalue for T.

   By definition, the transformation $T^2 - \lambda^2 I$ has a nontrivial kernel. Let $x \in \ker(T^2 - \lambda^2 I)$. Then $T^2 - \lambda^2 I = (T - \lambda I)(T + \lambda I)$, not as multiplication but as function composition. Then either $x \in \ker(T + \lambda I)$ or $(T + \lambda I)(x) \in \ker(T - \lambda I)$. In the first case, then $-\lambda$ is an eigenvalue for x of T. In the second case, we can assume that $x \notin \ker(T + \lambda I)$. Then λ is an eigenvalue for T corresponding to the vector $(T + \lambda I)(x) \neq 0$.

5. Let V be the linear space of all real functions differentiable on $(0, 1)$. If $f \in V$, define $T(f)(t) = tf'(t)$ for all $t \in (0, 1)$. Prove that every real λ is an eigenvalue for T, and determine the eigenfunctions corresponding to λ.

   An eigenfunction f for T corresponding to λ would satisfy the differential equation $tf'(t) = \lambda f(t)$. We can solve this by separation of variables so that $f(t) = t^\lambda$. Indeed this linear function can be seen to satisfy the desired diff eq. Note that t^λ is well-defined on $(0, 1)$ for all λ.

7. Let V be the linear space of all functions continuous on $(-\infty, \infty)$ and such that the integral $\int_{-\infty}^{x} f(t)\,dt$ exists for all real x. If $f \in V$, let $T(f)(x) = \int_{-\infty}^{x} f(t)\,dt$. Prove that every positive λ is an eigenvalue for T and determine the eigenfunctions corresponding to λ.

An eigenfunction for T would satisfy the relation

$$\int_{-\infty}^{x} f(t)\,dt = \lambda f(t).$$

Taking the derivative, the fundamental theorem of calculus states that f needs to satisfy

$$f(x) = \lambda f'(x).$$

Solving this diff eq by separation of variables, we see that $f(x) = e^{x/\lambda}$ is an eigenfunction for T corresponding to $\lambda > 0$.

11. Assume that a linear transformation T has two eigenvectors x and y belonging to distinct eigenvalues λ and μ. If $ax + by$ is an eigenvector of T, prove that $a = 0$ or $b = 0$.

Let $ax + by$ correspond to the eigenvalue γ. Then on the one hand

$$T(ax + by) = \gamma(ax + by).$$

On the other hand,

$$T(ax + by) = a\lambda x + b\mu y.$$

Therefore $\gamma(ax + by) = a\lambda x + b\mu y$. Note that since $\lambda \neq \mu$, then $x \notin \text{span}(y)$, so that x and y are independent. The previous equation can be turned into the relation

$$a(\gamma - \lambda)x + b(\gamma - \mu)y = 0.$$

By the definition of independence, if $a, b \neq 0$, then $\gamma = \lambda$ and $\gamma = \mu$. This is a contradiction, so that either $a = 0$ or $b = 0$.

4.8: 1acd, 2, 3, 7a, 8a, 10, 11, 14

1. (a) $\det \begin{bmatrix} \lambda - 1 & 0 \\ 0 & \lambda - 1 \end{bmatrix} = (\lambda - 1)^2$

   This matrix has characteristic polynomial $(\lambda - 1)^2$, so has eigenvalue 1. All nonzero vectors in $\mathbb{R}^2$ are eigenvectors corresponding to 1. The dimension of the eigenspace is 2.

   (c) $\det \begin{bmatrix} \lambda - 1 & 0 \\ -1 & \lambda - 1 \end{bmatrix} = (\lambda - 1)^2$

   The characteristic polynomial is $(\lambda - 1)^2$ so the only eigenvalue is 1. The eigenvectors are all nonzero vectors of the form $(0, t)$. The dimension of the eigenspace is 1.

   (d) $\det \begin{bmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{bmatrix} = (\lambda - 1)^2 - 1 = \lambda^2 - 2\lambda$

   The characteristic polynomial is $\lambda(\lambda - 2)$ so the eigenvalues are 0 and 2. The eigenvectors corresponding to 0 are all nonzero vectors of the form $(t, -t)$. The eigenvectors corresponding to 2 are all nonzero vectors of the form (t, t). Both eigenspaces are of dimension 1.

2. $\det \begin{bmatrix} \lambda - 1 & -a \\ -b & \lambda - 1 \end{bmatrix} = (\lambda - 1)^2 - ab = \lambda^2 - 2\lambda + 1 - ab$

   Using the quadratic formula, we find the eigenvalues:

$$\begin{aligned} \lambda &= \frac{2 \pm \sqrt{4 - 4(1 - ab)}}{2} \\ &= 1 \pm \sqrt{ab} \end{aligned}$$

   Since $a, b > 0$ these eigenvalues are always distinct real numbers. For $\lambda = 1 + \sqrt{ab}$, we have the following:

$$\begin{bmatrix} \sqrt{ab} & -a \\ -b & \sqrt{ab} \end{bmatrix} \longrightarrow \begin{bmatrix} \sqrt{b} & -\sqrt{a} \\ 0 & 0 \end{bmatrix}$$

   So we have eigenvectors of the form $(t\sqrt{a}, t\sqrt{b})$ where $t \neq 0$ and the eigenspace is of dimension 1.

   For $\lambda = 1 - \sqrt{ab}$, we have the following:

$$\begin{bmatrix} -\sqrt{ab} & -a \\ -b & -\sqrt{ab} \end{bmatrix} \longrightarrow \begin{bmatrix} \sqrt{b} & \sqrt{a} \\ 0 & 0 \end{bmatrix}$$

   So we have eigenvectors of the form $(t\sqrt{a}, -t\sqrt{b})$ where $t \neq 0$ and the eigenspace is of dimension 1.

3.

$$\det \begin{bmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{bmatrix} = (\lambda - \cos\theta)^2 + \sin^2\theta$$
$$= \lambda^2 - 2\lambda\cos\theta + \cos^2\theta + \sin^2\theta$$
$$= \lambda^2 - 2\lambda\cos\theta + 1$$

Using the quadratic formula, our eigenvalues are

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$
$$= \cos\theta \pm \sqrt{-\sin^2\theta}$$

First, assume we are working over $\mathbb{R}$. Then we only have real eigenvalues when $\sin\theta = 0$. For $\theta = 0$, we have the matrix I with repeated eigenvalue 1 and for $\theta = \pi$ we have the matrix $-I$ with repeated eigenvalue -1. In both cases, each nonzero vector in $\mathbb{R}^2$ is an eigenvector and the eigenspace has dimension 2.

Now assume we are working over $\mathbb{C}$. Then we have eigenvalues $\cos\theta \pm i\sin\theta$. If $\theta = 0$, we have the case analyzed above. If $\theta \neq 0$, we have distinct complex conjugate eigenvectors. For $\lambda = \cos\theta + i\sin\theta$, we have

$$\begin{bmatrix} i\sin\theta & \sin\theta \\ -\sin\theta & i\sin\theta \end{bmatrix} \longrightarrow \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}$$

So we have eigenvectors of the form (ti, t) where $t \neq 0$ and the eigenspace is of dimension 1. For $\lambda = \cos\theta - i\sin\theta$, we have

$$\begin{bmatrix} -i\sin\theta & \sin\theta \\ -\sin\theta & -i\sin\theta \end{bmatrix} \longrightarrow \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix}$$

So we have eigenvectors of the form $(-ti, t)$ where $t \neq 0$ and the eigenspace is of dimension 1.

7a. $\det \begin{bmatrix} \lambda - 1 & 0 & 0 \\ 3 & \lambda - 1 & 0 \\ -4 & 7 & \lambda - 1 \end{bmatrix} = (\lambda - 1)^3$

The characteristic polynomial is $(\lambda - 1)^3$ so we have repeated eigenvalue 1. To find the eigenvectors, we look at the following:

$$\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ -4 & 7 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have eigenvectors of the form $(0, 0, t)$ where $t \neq 0$ and an eigenspace of dimension 1.

8a.

$$\det\begin{bmatrix}\lambda & 0 & -1 & 0\\ 0 & \lambda & 0 & -1\\ -1 & 0 & \lambda & 0\\ 0 & -1 & 0 & \lambda\end{bmatrix} = \lambda\det\begin{bmatrix}\lambda & 0 & -1\\ 0 & \lambda & 0\\ -1 & 0 & \lambda\end{bmatrix} - \det\begin{bmatrix}0 & \lambda & -1\\ -1 & 0 & 0\\ 0 & -1 & \lambda\end{bmatrix}$$
$$= \lambda\left(\lambda\det\begin{bmatrix}\lambda & -1\\ -1 & \lambda\end{bmatrix}\right) - \left(\det\begin{bmatrix}\lambda & -1\\ -1 & \lambda\end{bmatrix}\right)$$
$$= \lambda^2(\lambda^2-1) - (\lambda^2-1)$$
$$= (\lambda-1)^2(\lambda+1)^2$$

This matrix has eigenvalues 1,1, -1, -1.

10. For any λ, we have $(\lambda I - A)^T = (\lambda I)^T - A^T = \lambda I - A^T$. Since the determinant is preserved by taking matrix transposes, $\det(\lambda I - A) = \det(\lambda I - A^T)$. That is, the characteristic polynomials of A and A^T are the same.

11. Let A, B be $n \times n$ matrices with A nonsingular. Suppose that λ is an eigenvalue of AB so that $\det(\lambda I - AB) = 0$. Since A is nonsingular, A^{-1} exists and both $\det A$ and $\det A^{-1}$ are nonzero. Note that

$$A^{-1}(\lambda I - AB)A = \lambda I - BA$$

so $\det(\lambda I - BA) = \det A^{-1}\det(\lambda I - AB)\det A = 0$. Thus λ is an eigenvalue of BA.

On the other hand, if λ is an eigenvalue of BA then $\det(\lambda I - BA) = 0$. Since we have $0 = \det(\lambda I - BA) = \det A^{-1}\det(\lambda I - AB)\det A$ and both $\det A$ and $\det A^{-1}$ are nonzero, $\det(\lambda I - AB) = 0$. So λ is an eigenvalue of AB.

14. (a) The diagonal elements of $A + B$ are $a_{ii} + b_{ii}$, so

$$\begin{aligned}\operatorname{tr}(A+B) &= \sum_{i=1}^{n}(a_{ii}+b_{ii})\\ &= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}\\ &= \operatorname{tr} A + \operatorname{tr} B\end{aligned}$$

(b) The diagonal elements of cA are ca_{ii}, so

$$\begin{aligned}\operatorname{tr}(cA) &= \sum_{i=1}^{n} ca_{ii}\\ &= c\sum_{i=1}^{n} a_{ii}\\ &= c\operatorname{tr}(A)\end{aligned}$$

(c) The diagonal elements of AB are $\sum_{k=1}^{n} a_{ik}b_{ki}$ and the diagonal elements of BA are $\sum_{k=1}^{n} b_{ik}a_{ki}$. So we have

$$\begin{aligned}
\operatorname{tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}b_{ki} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki}a_{ik} \\
&= \sum_{k=1}^{n} (BA)_{kk} \\
&= \operatorname{tr}(BA)
\end{aligned}$$

(d) The diagonal elements of A^T are a_{ii} so $\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii} = \operatorname{tr}(A^T)$.

Additional Problems.

I. A matrix $A = (a_{ij})$ is upper triangular if $a_{ij} = 0$ whenever $i > j$. Using expansion by minors, show that the determinant of an upper-triangular matrix is the product of its diagonal entries.

We proceed by induction. If A is 1×1, then A is always upper triangular and $\det A$ is the single (necessarily diagonal) element.

Suppose we have shown this for $(n-1) \times (n-1)$ matrices and that A is $n \times n$ and upper triangular. We expand $\det A$ down the first column. The only nonzero entry in that column is a_{11}, so $\det A = (-1)^{1+1} a_{11} \det A_{11} = a_{11} \det A_{11}$. Note that the minor matrix A_{11} is upper triangular and has diagonal elements $a_{22}, a_{33}, \ldots, a_{nn}$. By induction, $\det A_{11}$ is the product of these diagonal elements and so $\det A = a_{11} a_{22} \cdots a_{nn}$ as desired.

II. Suppose V is a finite-dimensional vector space and $0 \neq x_0 \in V$. Show there is a linear map $f : V \to \mathbb{R}$ so that $f(x_0) = 1$. Is f unique?

Extend the independent set $\{x_0\}$ to a basis $\{x_0, x_1, \ldots, x_k\}$ of V. Then define $f : V \to \mathbb{R}$ by $f(x_0) = 1$ and $f(x_i) = 0$ for $i > 0$. Note that f is linear.

If $\dim V > 1$, this map is not unique since we could take instead $g : V \to \mathbb{R}$ where $g(x_0) = 1$ and $g(x_i) = 1$ for $i > 0$.

If $\dim V = 1$, the set $\{x_0\}$ forms a basis of V and since any linear transformation is completely determined by its values on the basis elements, f is determined by $f(x_0)$. So in this case f is unique.

III. Suppose V is a finite-dimensional vector space and $W \subseteq V$ is a fixed subspace. Prove there is a linear map defined on V with kernel exactly W.

Fix a basis $\{w_1, \ldots, w_k\}$ of W and extend it to a basis $\{w_1, \ldots, w_k, v_1, \ldots, v_n\}$ of V. Define $T : V \to V$ by $T(w_i) = \vec{0}$ and $T(v_j) = v_j$. Note that T is linear. Clearly $W \subset \ker T$.

Let $v \in \ker T$ be arbitrary. Since $v \in V$, we can write

$$v = \sum_{i=1}^{k} c_i w_i + \sum_{j=1}^{n} d_j v_j$$

for some choice of constants c_i and d_j. Applying T, we have

$$\begin{aligned} T(v) = \vec{0} &= \sum_{i=1}^{k} c_i T(w_i) + \sum_{j=1}^{n} d_j T(v_j) \\ &= \sum_{i=1}^{k} c_i \vec{0} + \sum_{j=1}^{n} d_j v_j \\ &= \sum_{j=1}^{n} d_j v_j \end{aligned}$$

Since the vectors $\{v_1, \ldots, v_n\}$ are part of a basis, they are linearly independent and so each $d_j = 0$. Thus $v = \sum_{i=1}^{k} c_i w_i \in W$. So in fact $W = \ker T$ as desired.