1. (a) If $T$ has an eigenvalue $\lambda$, prove that $aT$ has the eigenvalue $a\lambda$.

Let $x$ be an eigenvector of $T$ corresponding to $\lambda$. We show that $(aT)(x) = (a\lambda)x$.

Indeed

$$(aT)(x) = aT(x) = a\lambda x = (a\lambda)x.$$ 

This proves the claim.

(b) If $x$ is an eigenvector for both $T_1$ and $T_2$, prove that $x$ is also an eigenvector for $aT_1 + bT_2$. How are the eigenvalues related?

Let $x$ be an eigenvector corresponding to $\lambda_1$ and $\lambda_2$ (for $T_1$ and $T_2$ respectively).

We claim that $x$ is an eigenvector corresponding to $\lambda = a\lambda_1 + b\lambda_2$. Indeed

$$(aT_1 + bT_2)(x) = aT_1(x) + bT_2(x) = a\lambda_1 x + b\lambda_2 x = (a\lambda_1 + b\lambda_2)x.$$ 

4. If $T: V \to V$ has the property that $T^2$ has a nonnegative eigenvalue $\lambda^2$, prove that at least one of $\lambda$ or $-\lambda$ is an eigenvalue for $T$.

By definition, the transformation $T^2 - \lambda^2 I$ has a nontrivial kernel. Let $x \in \ker (T^2 - \lambda^2 I)$. Then $T^2 - \lambda^2 I = (T - \lambda I)(T + \lambda I)$, not as multiplication but as function composition. Then either $x \in \ker (T + \lambda I)$ or $(T + \lambda I)(x) \in \ker (T - \lambda I)$. In the first case, then $-\lambda$ is an eigenvalue for $x$ of $T$. In the second case, we can assume that $x \notin \ker (T + \lambda I)$. Then $\lambda$ is an eigenvalue for $T$ corresponding to the vector $(T + \lambda I)(x) \neq 0$.

5. Let $V$ be the linear space of all real functions differentiable on $(0, 1)$. If $f \in V$, define $T(f)(t) = tf'(t)$ for all $t \in (0, 1)$. Prove that every real $\lambda$ is an eigenvalue for $T$, and determine the eigenfunctions corresponding to $\lambda$.

An eigenfunction $f$ for $T$ corresponding to $\lambda$ would satisfy the differential equation $tf'(t) = \lambda f(t)$. We can solve this by separation of variables so that $f(t) = t^\lambda$. Indeed this linear function can be seen to satisfy the desired diff eq. Note that $t^\lambda$ is well-defined on $(0, 1)$ for all $\lambda$.

7. Let $V$ be the linear space of all functions continuous on $(-\infty, \infty)$ and such that the integral $\int_{-\infty}^{x} f(t) dt$ exists for all real $x$. If $f \in V$, let $T(f)(x) = \int_{-\infty}^{x} f(t) dt$. Prove that every positive $\lambda$ is an eigenvalue for $T$ and determine the eigenfunctions corresponding to $\lambda$. 

1
An eigenfunction for $T$ would satisfy the relation
\[
\int_{-\infty}^{x} f(t) \, dt = \lambda f(x).
\]
Taking the derivative, the fundamental theorem of calculus states that $f$ needs to satisfy
\[
f(x) = \lambda f'(x).
\]
Solving this diff eq by separation of variables, we see that $f(x) = e^{x/\lambda}$ is an eigenfunction for $T$ corresponding to $\lambda > 0$.

11. Assume that a linear transformation $T$ has two eigenvectors $x$ and $y$ belonging to distinct eigenvalues $\lambda$ and $\mu$. If $ax + by$ is an eigenvector of $T$, prove that $a = 0$ or $b = 0$.

Let $ax + by$ correspond to the eigenvalue $\gamma$. Then on the one hand
\[
T(ax + by) = \gamma(ax + by).
\]
On the other hand,
\[
T(ax + by) = a\lambda x + b\mu y.
\]
Therefore $\gamma(ax + by) = a\lambda x + b\mu y$. Note that since $\lambda \neq \mu$, then $x \notin \text{span}(y)$, so that $x$ and $y$ are independent. The previous equation can be turned into the relation
\[
a(\gamma - \lambda)x + b(\gamma - \mu)y = 0.
\]
By the definition of independence, if $a, b \neq 0$, then $\gamma = \lambda$ and $\gamma = \mu$. This is a contradiction, so that either $a = 0$ or $b = 0$. 

2
1. (a) \[
\begin{bmatrix}
\lambda - 1 & 0 \\
0 & \lambda - 1
\end{bmatrix}
= (\lambda - 1)^2
\]
This matrix has characteristic polynomial \((\lambda - 1)^2\), so has eigenvalue 1. All nonzero vectors in \(\mathbb{R}^2\) are eigenvectors corresponding to 1. The dimension of the eigenspace is 2.

(c) \[
\begin{bmatrix}
\lambda - 1 & 0 \\
-1 & \lambda - 1
\end{bmatrix}
= (\lambda - 1)^2
\]
The characteristic polynomial is \((\lambda - 1)^2\) so the only eigenvalue is 1. The eigenvectors are all nonzero vectors of the form \((0, t)\). The dimension of the eigenspace is 1.

(d) \[
\begin{bmatrix}
\lambda - 1 & -1 \\
-1 & \lambda - 1
\end{bmatrix}
= (\lambda - 1)^2 - 1 = \lambda^2 - 2\lambda
\]
The characteristic polynomial is \(\lambda(\lambda - 2)\) so the eigenvalues are 0 and 2. The eigenvectors corresponding to 0 are all nonzero vectors of the form \((t, -t)\). The eigenvectors corresponding to 2 are all nonzero vectors of the form \((t, t)\). Both eigenspaces are of dimension 1.

2. \[
\begin{bmatrix}
\lambda - 1 & -a \\
-b & \lambda - 1
\end{bmatrix}
= (\lambda - 1)^2 - ab = \lambda^2 - 2\lambda + 1 - ab
\]
Using the quadratic formula, we find the eigenvalues:
\[
\lambda = \frac{2 \pm \sqrt{4 - 4(1 - ab)}}{2}
= 1 \pm \sqrt{ab}
\]
Since \(a, b > 0\) these eigenvalues are always distinct real numbers. For \(\lambda = 1 + \sqrt{ab}\), we have the following:
\[
\begin{bmatrix}
\sqrt{ab} & -a \\
-b & \sqrt{ab}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\sqrt{b} & -\sqrt{a} \\
0 & 0
\end{bmatrix}
\]
So we have eigenvectors of the form \((t\sqrt{a}, t\sqrt{b})\) where \(t \neq 0\) and the eigenspace is of dimension 1.

For \(\lambda = 1 - \sqrt{ab}\), we have the following:
\[
\begin{bmatrix}
-\sqrt{ab} & -a \\
-b & -\sqrt{ab}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\sqrt{b} & \sqrt{a} \\
0 & 0
\end{bmatrix}
\]
So we have eigenvectors of the form \((t\sqrt{a}, -t\sqrt{b})\) where \(t \neq 0\) and the eigenspace is of dimension 1.
3. 

\[
\begin{vmatrix}
\lambda - \cos \theta & \sin \theta \\
-\sin \theta & \lambda - \cos \theta
\end{vmatrix}
= (\lambda - \cos \theta)^2 + \sin^2 \theta
\]

\[
= \lambda^2 - 2\lambda \cos \theta + \cos^2 \theta + \sin^2 \theta
\]

\[
= \lambda^2 - 2\lambda \cos \theta + 1
\]

Using the quadratic formula, our eigenvalues are 

\[
\lambda = \frac{2 \cos \theta \pm \sqrt{4 \cos^2 \theta - 4}}{2}
\]

\[
= \cos \theta \pm \sqrt{-\sin^2 \theta}
\]

First, assume we are working over \( \mathbb{R} \). Then we only have real eigenvalues when \( \sin \theta = 0 \). For \( \theta = 0 \), we have the matrix \( I \) with repeated eigenvalue 1 and for \( \theta = \pi \) we have the matrix \( -I \) with repeated eigenvalue \(-1\). In both cases, each nonzero vector in \( \mathbb{R}^2 \) is an eigenvector and the eigenspace has dimension 2.

Now assume we are working over \( \mathbb{C} \). Then we have eigenvalues \( \cos \theta \pm i \sin \theta \). If \( \theta = 0 \), we have the case analyzed above. If \( \theta \neq 0 \), we have distinct complex conjugate eigenvectors. For \( \lambda = \cos \theta + i \sin \theta \), we have

\[
\begin{bmatrix}
i \sin \theta & \sin \theta \\
-\sin \theta & i \sin \theta
\end{bmatrix}
\rightarrow
\begin{bmatrix}i & 1 \\
0 & 0
\end{bmatrix}
\]

So we have eigenvectors of the form \((ti, t)\) where \(t \neq 0\) and the eigenspace is of dimension 1. For \( \lambda = \cos \theta - i \sin \theta \), we have

\[
\begin{bmatrix}
-i \sin \theta & \sin \theta \\
-\sin \theta & -i \sin \theta
\end{bmatrix}
\rightarrow
\begin{bmatrix}
-i & 1 \\
0 & 0
\end{bmatrix}
\]

So we have eigenvectors of the form \((-ti, t)\) where \(t \neq 0\) and the eigenspace is of dimension 1.

7a. 

\[
\begin{vmatrix}
\lambda - 1 & 0 & 0 \\
3 & \lambda - 1 & 0 \\
-4 & 7 & \lambda - 1
\end{vmatrix}
= (\lambda - 1)^3
\]

The characteristic polynomial is \((\lambda - 1)^3\) so we have repeated eigenvalue 1. To find the eigenvectors, we look at the following:

\[
\begin{bmatrix}0 & 0 & 0 \\
3 & 0 & 0 \\
-4 & 7 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

So we have eigenvectors of the form \((0, 0, t)\) where \(t \neq 0\) and an eigenspace of dimension 1.
8a.
\[
\det \begin{bmatrix}
\lambda & 0 & -1 & 0 \\
0 & \lambda & 0 & -1 \\
-1 & 0 & \lambda & 0 \\
0 & -1 & 0 & \lambda
\end{bmatrix}
= \lambda \det \begin{bmatrix}
\lambda & 0 & -1 \\
0 & \lambda & 0 \\
-1 & 0 & \lambda
\end{bmatrix}
- \det \begin{bmatrix}
0 & \lambda & -1 \\
-1 & 0 & 0 \\
0 & -1 & \lambda
\end{bmatrix}
= \lambda \left( \lambda \det \begin{bmatrix}
\lambda & -1 \\
-1 & \lambda
\end{bmatrix} \right) - \left( \det \begin{bmatrix}
\lambda & -1 \\
-1 & \lambda
\end{bmatrix} \right)
= \lambda^2 (\lambda^2 - 1) - (\lambda^2 - 1)
= (\lambda - 1)^2 (\lambda + 1)^2
\]

This matrix has eigenvalues 1,1, -1, -1.

10. For any \( \lambda \), we have \((\lambda I - A)^T = (\lambda I)^T - A^T = \lambda I - A^T\). Since the determinant is preserved by taking matrix transposes, \(\det(\lambda I - A) = \det(\lambda I - A^T)\). That is, the characteristic polynomials of \(A\) and \(A^T\) are the same.

11. Let \(A, B\) be \(n \times n\) matrices with \(A\) nonsingular. Suppose that \(\lambda\) is an eigenvalue of \(AB\) so that \(\det(\lambda I - AB) = 0\). Since \(A\) is nonsingular, \(A^{-1}\) exists and both \(\det A\) and \(\det A^{-1}\) are nonzero. Note that
\[
A^{-1}(\lambda I - AB)A = \lambda I - BA
\]
so \(\det(\lambda I - BA) = \det A^{-1} \det(\lambda I - AB) \det A = 0\). Thus \(\lambda\) is an eigenvalue of \(BA\).

On the other hand, if \(\lambda\) is an eigenvalue of \(BA\) then \(\det(\lambda I - BA) = 0\). Since we have \(0 = \det(\lambda I - BA) = \det A^{-1} \det(\lambda I - AB) \det A\) and both \(\det A\) and \(\det A^{-1}\) are nonzero, \(\det(\lambda I - AB) = 0\). So \(\lambda\) is an eigenvalue of \(AB\).

14. (a) The diagonal elements of \(A + B\) are \(a_{ii} + b_{ii}\), so
\[
\text{tr}(A + B) = \sum_{i=1}^{n} (a_{ii} + b_{ii})
= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}
= \text{tr} A + \text{tr} B
\]
(b) The diagonal elements of \(cA\) are \(ca_{ii}\), so
\[
\text{tr}(cA) = \sum_{i=1}^{n} ca_{ii}
= c \sum_{i=1}^{n} a_{ii}
= c \text{tr}(A)
\]
(c) The diagonal elements of $AB$ are $\sum_{k=1}^{n} a_{ik} b_{ki}$ and the diagonal elements of $BA$ are $\sum_{k=1}^{n} b_{ik} a_{ki}$. So we have

$$\text{tr}(AB) = \sum_{i=1}^{n} (AB)_{ii}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki}$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki} a_{ik}$$

$$= \sum_{k=1}^{n} (BA)_{kk}$$

$$= \text{tr}(BA)$$

(d) The diagonal elements of $A^T$ are $a_{ii}$ so $\text{tr}(A) = \sum_{i=1}^{n} a_{ii} = \text{tr}(A^T)$.
Additional Problems.

I. A matrix $A = (a_{ij})$ is upper triangular if $a_{ij} = 0$ whenever $i > j$. Using expansion by minors, show that the determinant of an upper-triangular matrix is the product of its diagonal entries.

We proceed by induction. If $A$ is $1 \times 1$, then $A$ is always upper triangular and $\det A$ is the single (necessarily diagonal) element.

Suppose we have shown this for $(n-1) \times (n-1)$ matrices and that $A$ is $n \times n$ and upper triangular. We expand $\det A$ down the first column. The only nonzero entry in that column is $a_{11}$, so $\det A = (-1)^{1+1} a_{11} \det A_{11} = a_{11} \det A_{11}$. Note that the minor matrix $A_{11}$ is upper triangular and has diagonal elements $a_{22}, a_{33}, \ldots, a_{nn}$. By induction, $\det A_{11}$ is the product of these diagonal elements and so $\det A = a_{11}a_{22} \cdots a_{nn}$ as desired.

II. Suppose $V$ is a finite-dimensional vector space and $0 \neq x_0 \in V$. Show there is a linear map $f : V \to \mathbb{R}$ so that $f(x_0) = 1$. Is $f$ unique?

Extend the independent set $\{x_0\}$ to a basis $\{x_0, x_1, \ldots, x_k\}$ of $V$. Then define $f : V \to \mathbb{R}$ by $f(x_0) = 1$ and $f(x_i) = 0$ for $i > 0$. Note that $f$ is linear.

If $\dim V > 1$, this map is not unique since we could take instead $g : V \to \mathbb{R}$ where $g(x_0) = 1$ and $g(x_i) = 1$ for $i > 0$.

If $\dim V = 1$, the set $\{x_0\}$ forms a basis of $V$ and since any linear transformation is completely determined by its values on the basis elements, $f$ is determined by $f(x_0)$. So in this case $f$ is unique.

III. Suppose $V$ is a finite-dimensional vector space and $W \subseteq V$ is a fixed subspace. Prove there is a linear map defined on $V$ with kernel exactly $W$.

Fix a basis $\{w_1, \ldots, w_k\}$ of $W$ and extend it to a basis $\{w_1, \ldots, w_k, v_1, \ldots, v_{n}\}$ of $V$. Define $T : V \to V$ by $T(w_i) = \vec{0}$ and $T(v_j) = v_j$. Note that $T$ is linear. Clearly $W \subset \ker T$.

Let $v \in \ker T$ be arbitrary. Since $v \in V$, we can write

$$v = \sum_{i=1}^{k} c_i w_i + \sum_{j=1}^{n} d_j v_j$$
for some choice of constants $c_i$ and $d_j$. Applying $T$, we have

$$T(v) = \vec{0} = \sum_{i=1}^{k} c_i T(w_i) + \sum_{j=1}^{n} d_j T(v_j)$$

$$= \sum_{i=1}^{k} c_i \vec{0} + \sum_{j=1}^{n} d_j v_j$$

$$= \sum_{j=1}^{n} d_j v_j$$

Since the vectors $\{v_1, \ldots, v_n\}$ are part of a basis, they are linearly independent and so each $d_j = 0$. Thus $v = \sum_{i=1}^{k} c_i w_i \in W$. So in fact $W = \ker T$ as desired.