(a) Find the gradient vector at each point at which it exists for the scalar fields defined by the following equations. (a) \( f(x, y) = x^2 + y^2 \sin(xy) \)  
(b) \( f(x, y, z) = x^2 y^3 z^4 \)  
(c) \( f(x, y, z) = \log(x^2 + 2y^2 - 3z^2) \).

(a) This is a composition, product, and sum of globally differentiable functions, and thus has gradient everywhere.

\[
\nabla f = (2x + y^3 \cos(xy), 2y \sin(xy) + xy^2 \cos(xy))
\]

(c) Similarly, this is differentiable everywhere and

\[
\nabla f = (2xy^3 z^4, 3x^2 y^2 z^4, 4x^2 y^3 z^3).
\]

(e) Since log is differentiable only for positive reals, then we must restrict the open set where \( x^2 + 2y^2 - 3z^2 > 0 \). On this domain we have that

\[
\nabla f = \frac{1}{x^2 + 2y^2 - 3z^2}(2x, 4y, -6z)
\]

3. Find the points \((x, y)\) and the directions for which the directional derivative of \( f(x, y) = 3x^2 + y^2 \) has its largest value, if \((x, y)\) is on the unit circle.

We can maximize the directional derivative at each point on the unit circle, and then find the points that achieve the maximum of that function.

The maximum directional derivative at a point \( x \) is achieved by the direction \( u = \frac{\nabla f}{||\nabla f||} \).

In this case the angle between the direction and the gradient is 0, therefore the dot product is maximized. In this case

\[
D_{\text{max}}(f) = \nabla f \cdot \frac{\nabla f}{||\nabla f||} = ||\nabla f||.
\]

For \( f(x, y) = 3x^2 + y^2 \), we obtain

\[
D_{\text{max}}(f) = ||(6x, 2y)|| = \sqrt{36x^2 + 4y^2} = 2\sqrt{9x^2 + y^2}.
\]

Thus we must maximize this function on the unit circle.

We parametrize the unit circle by \((\cos(t), \sin(t))\), so and rewrite \( f \) as \( g(t) = 2\sqrt{9\cos^2(t) + \sin^2(t)} = 2\sqrt{8\cos^2(t) + 1} \). We obtain the critical points by solving \( g'(t) = 0 \). The numerator of the derivative is \( \sin(t) \cos(t) \), so it suffices to solve \( \sin(t) \cos(t) = 0 \). This occurs at \( t = \frac{k\pi}{2} \) for \( k \in \mathbb{Z} \). Taking the double derivative, we obtain maxima at \( t = k\pi \) for \( k \in \mathbb{Z} \), i.e. angles of 0 and \( \pi \). This corresponds to the points \((1, 0)\) and \((-1, 0)\). The unit vector which gives the maximum directional derivative at the points is also \((1, 0)\) and \((-1, 0)\) by the above formula.
4. A differentiable scalar field \( f \) has at the points \((1,2)\) directional derivatives 2 in the direction toward \((2,2)\) and -2 in the direction toward \((1,1)\). Determine the gradient vector at \((1,2)\) and compute the directional derivative in the direction toward \((4,6)\).

Note that \((2,2) − (1,2) = (1,0)\) and \((1,1) − (1,2) = −(0,1)\). Therefore the assumptions are the problem is that \( D_1(f) = 2 \) and \( D_2(f) = 2 \). Therefore the gradient is \((2,2)\). Furthermore the directional derivative toward \((4,6)\) is the directional derivative of the vector

\[
\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||}
\]

where \( \mathbf{v} = (4,6) − (1,2) = (3,4) \). Therefore \( u = (3/5, 4/5) \) and \( D_u(f) = (2,2) \cdot u = 14/5 \).

9. Assume \( f \) is differentiable at each point of an \( n \)-ball \( B(a) \). If \( f'(x; y) = 0 \) for \( n \) independent vectors \( y_1, \ldots, y_n \) for every \( x \) in \( B(a) \), prove that \( f \) is constant on \( B(a) \).

It suffices to show that \( Df = 0 \) for all \( x \in B(a) \). Since \( f \) is differentiable, then \( f'(x; y) = \nabla f \cdot y \) exists for all \( y \). Write \( y = \sum_i a_i y_i \). Therefore,

\[
f'(x; y) = \nabla f \cdot y = \sum_i a_i \nabla f \cdot y_i = \sum_i a_i f'(x; y_i) = 0.
\]

Thus \( f \) is constant on \( B(a) \). In the case where \( y = e_i \), this says that \( \frac{\partial f}{\partial x_i} = 0 \) for all \( i \).

10. Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be differentiable in an \( n \)-ball \( B(a) \). Show that (a) if \( \nabla f = 0 \) for every \( x \in B(a) \), then \( f \) is constant on \( B(a) \). (b) If \( f(x) \leq f(a) \) for all \( x \in B(a) \), then \( \nabla f(a) = 0 \).

(a) Let \( x \in B(a) \). We show that \( f(x) = f(a) \). Since \( f \) is differentiable, then \( f'(a; v) \) exists. Then by the MVT for derivatives of scalar fields, Theorem 8.4, we have that there exists a \( t \in [0, 1] \) such that \( f(a + (x - a)) - f(a) = f'(a + t(x - a); (x - a)) \). Since \( a + t(x - a) \in B(a) \), then the RHS is 0, so that \( f(a) = f(x) \).

(b) We show that \( f'(a; e_i) = 0 \) for all \( i \). By definition,

\[
f'(a; e_i) = \lim_{h \rightarrow 0} \frac{f(a + he_i) - f(a)}{h}.
\]

Note that for sufficiently small \( h \), the vector \( a + he_i \in B(a) \) so that \( f(a + he_i) \leq f(a) \). Therefore this is a limit of nonpositive values and if the limit exists, it must be a nonpositive as well. Since \( f \) is differentiable at \( a \), this limit exists so that \( f'(a; e_i) \leq 0 \). Similarly, \( f'(a; e_i) = \lim_{h \rightarrow 0} \frac{f(a) - f(a + he_i)}{h} \), which by a similar argument must also be \( \leq 0 \). Therefore \( f'(a; e_i) = 0 \) and \( \nabla f(a) = 0 \).
1. Let \( u = f(x, y) \). Set \( x = x(t) \) and \( y = y(t) \). Then \( u = F(t) \). (a) Use the chain rule to compute \( F' \). (b) Similarly, compute \( F''(t) \).

(a) Let \( T(t) = (x(t), y(t)) \) so that \( F = f \circ T \). By the chain rule

\[
DF = Df(T(t))DT(t) = \left[ \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \right] \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t).
\]

(b) To compute the double derivative, it suffices to compute \( D(D_1f) \) and \( D(D_2f) \). By similar calculations to above:

\[
D\left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} x' + \frac{\partial^2 f}{\partial x \partial y} y',
\]

\[
D\left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} x' + \frac{\partial^2 f}{\partial y^2} y'.
\]

Now by the product rule:

\[
F''(t) = D(D_1f x') + D(D_2f y')
= D(D_1f) x' + x'' D_1f + y'' D_2f + y' D(D_2f)
= \left( \frac{\partial^2 f}{\partial x^2} x' + \frac{\partial^2 f}{\partial x \partial y} y' \right) x' + x'' \frac{\partial f}{\partial x} + y'' \frac{\partial f}{\partial y} + y' \left( \frac{\partial^2 f}{\partial x \partial y} x' + \frac{\partial^2 f}{\partial y^2} y' \right)
= x'' \frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x^2} (x')^2 + 2x' y' \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} (y')^2 + y'' \frac{\partial f}{\partial y}.
\]

2a. Use the previous exercise to compute the derivative of \( f(x(t), y(t)) = x(t)^2 + y(t)^2 = t^2 + t^4 \).

\[
F'(t) = (2t)(1) + (2t^2)(2t) = 2t + 4t^3 \quad \text{and} \quad F''(t) = (0)(2t) + (2)(1)^2 + 2(1)(2t)(0) + (2)(2t)^2 + (2)(2t^2) = 2 + 12t^2
\]

3. Evaluate some directional derivatives.

(a) Note that on a sphere, the normal vector is just the vector itself. Thus

\[
f'((2, 2, 1); (2, 2, 1)) = (3, -5, 2) \cdot (2, 2, 1) = 6 - 10 + 2 = -2.
\]

If we require the vector be a unit vector, divide it by 3.
(b) Similarly, for \( v \) on the sphere of radius 2, we can compute \( f'(v; v) = (2x, -2y, 0) \cdot v = 2(x^2 - y^2). \) If the normal vector needs to be a unit vector, then we obtain \( f'(v; v/2) = x^2 - y^2. \)

(c) First we find the intersection of the two curves. If \( z^2 = x^2 + y^2 \) and \( z^2 = 2x^2 + 2y^2 - 25 \) then on the intersection, we have \( x^2 + y^2 = 25 \), which is circle of radius 5, and therefore \( z = 5 \) as well. Thus a parametrization of this intersection is \( \phi(t) = (5 \cos(t), 5 \sin(t), 5). \) The tangent vector is \( \phi'(t) = (-5 \sin(t), 5 \cos(t), 0) \), and at \((3, 4, 5)\), we obtain that the tangent vector is \( v = (-4, 3, 0) \). Normalizing, \( v = (-4/5, 3/5, 0). \) Now

\[
f'(((3, 4, 5); (-4/5, 3/5, 0)) = (6, 8, -10) \cdot (-4/5, 3/5, 0) = 0.
\]

4. (a) Find a vector \( V(x, y, z) \) normal to the surface \( z = ||(x, y, 0)|| + ||(x, y, 0)||^3. \) (b) Find the cosine of the angle \( \theta \) between \( V \) and the \( z \)-axis and determine the limit \( \cos(\theta) \) as \((x, y, z) \to (0, 0, 0)\).

The normal in this case is \( n = (-D_x f, -D_y f, 1) \) since we are on the graph of a scalar field. Therefore

\[
n = V(x, y, z) = -\left(\frac{x}{\sqrt{x^2 + y^2}} + 3x\sqrt{x^2 + y^2}, \frac{y}{\sqrt{x^2 + y^2}} + 3y\sqrt{x^2 + y^2}, -1\right).
\]

The cosine of the angle is \( \cos(\theta) = \frac{\langle v, w \rangle}{||v|| ||w||} \), so that

\[
\lim_{(x, y, z) \to (0, 0, 0)} \cos(\theta) = \lim_{(x, y) \to (0, 0)} \frac{1}{||V||} = \lim_{(x, y) \to (0, 0)} \frac{1}{\sqrt{1 + (1 + 3(x^2 + y^2)^2)}} = \frac{1}{\sqrt{2}}.
\]

6. Let \( f(x, y) = \sqrt{|xy|}. \) (a) Show that \( D_x f(0, 0) = D_y f(0, 0) = 0 \) and (b) determine whether the surface \( z = f(x, y) \) has a tangent plane at the origin. (a) By definition \( f'(0; (x, y)) = \lim_{h \to 0} \frac{\sqrt{|h^2 xy|}}{h} xy. \) For \((x, y) = e_i, \) we see that the limit is 0.

(b) To show that \( z = f(x, y) \) has no tangent plane, we show that \( f \) is not differentiable at 0. Indeed if we consider the directional derivative along \((1, 1)\), then

\[
f'(0; (1, 1)) = \lim_{h \to 0} \frac{\sqrt{|h^2|}}{h}
\]

which does not exist (since \( h \) can be positive or negative). In particular \( f'(0; (1, 1)) \neq \nabla f \cdot (1, 1). \) Thus \( f \) is not differentiable at 0 so there is no tangent plane.

12. If \( \nabla f \) is always parallel to \((x, y, z)\) show that \( f \) must assume equal values at the points \((0, 0, a)\) and \((0, 0, -a)\).
Let $\phi(t) = (0, a\sin(t), a\cos(t))$ for $t \in [-1, 1]$. Then by the chain rule,

$$D(f \circ \phi) = Df(\phi(t))D\phi(t) = \nabla f(\phi(t)) \begin{bmatrix} 0 \\ a\cos(t) \\ -a\sin(t) \end{bmatrix}.$$

But $\nabla f(\phi(t)) = \lambda(t)\phi(t)$, so that

$$D(f \circ \phi) = \lambda(t)\phi \cdot \phi' = 0.$$ 

Therefore $f$ has constant value on $\phi$, which implies that $f$ is equivalued on $(0, 0, a)$ and $(0, 0, -a)$. 
A function $G: \mathbb{R}^n \to \mathbb{R}$ is called homogeneous of degree 1 if $G(tx) = tG(x)$ for all $t > 0$ and all $x \neq 0 \in \mathbb{R}^n$. Suppose $G$ is homogeneous of degree 1 and continuous. (a) Show that $G(0) = 0$. (b) Show that the directional derivative of $G$ at 0 along $y$ exists for all $y \in \mathbb{R}^n$. (c) Show that $G$ is differentiable at 0 iff it is linear.

(a) Since $G(tx) = tG(x)$ for all $t > 0$ and $||x|| > 0$, we can fix $x \neq 0$ and consider

$$\lim_{t \to 0} G(tx) = \lim_{t \to 0} tG(x).$$

Taking the limit on the left hand side, we obtain that the limit is 0 since $G(x)$ is constant with respect to $t$. Taking the limit on the right side, by continuity of $G$, we get $G(0)$.

(b) Let $v \neq 0$, then we show that $G'(0; v)$ exists by definition.

$$G'(0; v) = \lim_{h \to 0} \frac{G(0 + hv) - G(0)}{h}$$

$$= \lim_{h \to 0} \frac{G(hv)}{h}$$

$$= \lim_{h \to 0} \frac{hG(v)}{h}$$

$$= G(v)$$

Hence we conclude $G'(0; v) = G(v)$.

(c) Assume $G$ is differentiable. Then $G(v) = G'(0; v) = DG(0)v$, which is a linear function. If $G$ is linear, then it is differentiable trivially.