Temporada 6-9

We want to prove that the inequality
\[ \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{3}{2} \]
holds for all positive real numbers \( a, b, c \).

**Proof:**

To prove the inequality, we will use the Cauchy-Schwarz inequality. Consider the vectors \( \mathbf{u} = (\sqrt{\frac{a}{b}}, \sqrt{\frac{b}{c}}, \sqrt{\frac{c}{a}}) \) and \( \mathbf{v} = (\sqrt{\frac{b}{a}}, \sqrt{\frac{c}{b}}, \sqrt{\frac{a}{c}}) \).

By the Cauchy-Schwarz inequality,
\[ \left( \sum_{cyc} \frac{a}{b} \right) \left( \sum_{cyc} \frac{b}{c} \right) \geq \left( \sum_{cyc} \frac{1}{2} \right)^2 \]

which simplifies to
\[ \sum_{cyc} \frac{a}{b} + \sum_{cyc} \frac{b}{c} \geq \frac{9}{2} \sum_{cyc} \frac{1}{a+b} \]

Since \( \sum_{cyc} \frac{a}{b} + \sum_{cyc} \frac{b}{c} = \frac{9}{2} \), we have
\[ \frac{9}{2} \geq \frac{9}{2} \sum_{cyc} \frac{1}{a+b} \]

which implies
\[ \sum_{cyc} \frac{1}{a+b} \leq \frac{1}{2} \]

Equality holds when \( a = b = c \).

**Strategy:**

Use the Cauchy-Schwarz inequality to prove the given inequality. The key is to recognize that the equality condition is achieved when the vectors are proportional.