Spring 2014

Problem S14#1. Prove that there is no one-to-one conformal map of the punctured disc \( \{ z \in \mathbb{C} : 0 < |z| < 1 \} \) onto the annulus \( \{ z \in \mathbb{C} : 1 < |z| < 2 \} \).

Theorem (Open Mapping). Let \( f \) be analytic on some open set \( V \subseteq \mathbb{C} \). If \( W \subseteq V \) is open and connected, then \( f(W) \) is open in \( \mathbb{C} \) or \( f \) is constant.

Solution. Let \( D \) be the open unit disc and let \( A \) be the given annulus. Suppose that there does exist a conformal bijection \( f : D \setminus \{ 0 \} \rightarrow A \). Since \( |f(z)| < 2 \), we have that
\[
\lim_{z \to 0} zf(z) = 0
\]
so \( f \) has a removable singularity at 0. Thus, there exists an analytic \( g : D \rightarrow \mathbb{C} \) such that \( g|_{D \setminus \{ 0 \}} = f \). Since \( f \) is continuous (from analytic, conformal... etc.) and because 0 is a limit point of \( D \setminus \{ 0 \} \), we know that \( g(0) \) must be a limit point of \( A \), i.e. \( g(0) \in \overline{A} \).

Since \( f \) is injective, \( g \) is not constant. By the Open Mapping Theorem \( g(D) \) is open. Since we also know \( g(D) \subseteq \overline{A} \), we must have that \( g(D) \subseteq \text{Int}(\overline{A}) = A \). So, let \( w = g(0) \in A \). Since \( f \) is surjective, there exists \( z \in D \setminus \{ 0 \} \) such that \( f(z) = w \). Let \( V, W \subset D \) be disjoint neighborhoods of \( z, 0 \) respectively. The Open Mapping Theorem implies that both \( g(V) \) and \( g(W) \) must be open, so \( g(V) \cap g(W) \) is open.

Since \( f \) is bijective, we know \( g(W) = \{ g(0) \} \cup g(W \setminus \{ 0 \}) \). So now
\[
g(V) \cap g(W) = g(V) \cap (\{ g(0) \} \cup g(W \setminus \{ 0 \}))
= f(V) \cap (\{ w \} \cup f(W \setminus \{ 0 \}))
= (f(V) \cap \{ w \}) \cup (f(V) \cap f(W \setminus \{ 0 \})).
\]
Since $f$ is bijective, $f(V) \cap f(W \setminus \{0\})$ is empty. So now

$$g(V) \cap g(W) = f(V) \cap \{w\} = \{w\}$$

which contradicts the openness of this intersection (from above).

OR

Since $g(V) \cap g(W)$ is a nonempty open subset of $A$, there is some $p \in g(V) \cap g(W)$ with $p \neq w$. But then there must exist two distinct $z_1 \in V$ and $z_2 \in W$ such that

$$f(z_1) = g(z_1) = p = g(z_2) = f(z_2).$$

Of course, this contradicts the bijectivity of $f$. □

**Problem S14#7.** Suppose $f(z)$ is analytic on the unit disc $D(0,1)$ and continuous on the closed unit disc $\overline{D}(0,1)$. Assume that $f(z) = 0$ on an arc of the circle $z = 1$. Show that $f(z) \equiv 0$.

**Theorem (Schwarz Reflection for analytic functions).** Let $U^+$ be an open set in the upper half plane $\mathbb{H}$ and suppose $\partial U^+$ contains an open interval of real numbers $I \subset \mathbb{R}$. Let $U^- = \{\overline{u} : u \in U\}$ be the reflection of $U^+$ across the real axis. Finally, let $U = I \cup U^+ \cup U^-$.

- Let $f$ be a function on $U$. If $f$ is analytic on $U \setminus I$ and $f$ is continuous and real-valued on $I$, then $f$ is analytic on $U$.

- Let $f$ be a function on $U^+ \cup I$. If $f$ is analytic on $U^+$ and $f$ is continuous and real-valued on $I$, then $f$ has a unique analytic continuation $F$ on $U$. Furthermore, it satisfies $F(z) = \overline{F(\overline{z})}$.

**Solution.** We will freely use the Möbius transformation $h: \mathbb{H} \to D$ defined by $h(z) = \frac{z-i}{z+i}$ which continuously maps the boundary $\partial \mathbb{H} = \mathbb{R}$ to the unit circle $\partial D$. Furthermore, we note that $h^{-1}$ exists and has the same continuous property on the boundary.

Note that the function $f \circ h$ is analytic on $\mathbb{H}$ and continuous on $\mathbb{R}$, furthermore it maps some interval $I \subset \mathbb{R}$ to 0. Let $\Omega^+ \subset \mathbb{H}$ be an open connected set with $\overline{\Omega^+} \cap \mathbb{R} = I$. Using the Schwarz
Reflection Principle, we obtain a function $F(z) = f \circ h(z)$ which is analytic on $\Omega = I \cup \Omega^+ \cup \Omega^-$ where $\Omega^-$ is the reflection of $\Omega^+$ over the real axis. Since $F(I) = \{0\}$, by the identity principle we know that in fact $F = 0$ on $\Omega$. Since $F|_H = f \circ h$, we must have $f \circ h(\Omega^+) = \{0\}$. By the identity principle again, $f \circ h$ must be zero on all of $\mathbb{H}$. Since $h$ is invertible, we have shown that $f(z) = 0$ for all $z \in \mathbb{T}$.

Problem S14#8. Show that every elliptic function $f$ of order $m$ has $m$ zeros in its fundamental parallelogram.

Theorem (Argument Principle). Let $f$ be a meromorphic function on an open connected set $\Omega$ with zeros at $z_j$ and poles at $p_k$. Let $\gamma \subset \Omega$ be a positively-oriented simple closed curve such that no zero and no pole of $f$ are on $\gamma$. Then

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z)}{f(z)} \, dz = \sum_j n(\gamma, z_j) - \sum_k n(\gamma, p_k)$$

where each $z_j$ and $p_k$ is repeated according to its multiplicity.

Solution. Just as in S13#8, we must assume that elliptic functions are non constant and that we count poles and zeros with their multiplicity. The order of $f$ is defined by the number of poles in its fundamental parallelogram. Let $w_1\mathbb{Z} + w_2\mathbb{Z}$ be the period lattice for $f$ and choose $a$ as a corner of a fundamental parallelogram $P$ such that no poles land on $\partial P$. We want to show that there are $m$ poles in $P$.

Since $f$ is elliptic, so is $f'$, and thus so is $\frac{f'(z)}{f(z)}$. Furthermore, $P$ is a fundamental parallelogram for $\frac{f'(z)}{f(z)}$. By the same argument as in S13#8, we know that

$$\int_{\partial P} \frac{f'(z)}{f(z)} \, dz = 0.$$

At the same time, we can apply the Argument Principle, to see that $\frac{1}{2\pi i} \int_{\partial P} \frac{f'(z)}{f(z)} \, dz = n_0 - n_p$ where $n_0$ is the number of zeros and $n_p$ is the number of poles inside $P$ (both counted with multiplicity, of course).
Problem F13#2. Let \( n \geq 2 \) be an integer. Evaluate the following integral:

\[
\int_{0}^{\infty} \frac{1}{1 + x^n} dx.
\]

Carefully justify all your steps.

Solution. Let \( f(z) = \frac{1}{1 + z^n} \), which has \( n \) simple poles at the \( n \)th roots of \(-1\), i.e. at every other of the \( 2n \)th roots of \( 1 \), specifically the set \( \left\{ \frac{\pi i}{n}, \frac{3\pi i}{n}, \frac{5\pi i}{n}, \ldots, \frac{(2n-1)\pi i}{n} \right\} \). Let \( \omega = e^{\frac{\pi i}{n}} \) be the primitive \( n \)th root of \(-1\). Let \( \gamma = I_R \cup \omega^2 I_R \cup C_R \) where \( I_R = [0, R] \in \mathbb{R} \) and \( C_R \) is the arc of the circle \(|z| = R\) from \( R \) to \( w^2 R \). By convention we orient \( \gamma \) counterclockwise as illustrated.

By the Residue Theorem, we know that

\[
\int_{\gamma} f(z) dz = 2\pi i \text{Res}_f(\omega) = 2\pi i \frac{1}{n\omega^{n-1}} = -2\pi i \frac{\omega}{R}.
\]

On the other hand, we know that

\[
\int_{\gamma} f(z) dz = \int_{I_R} f(z) dz + \int_{\omega^2 I_R} f(z) dz + \int_{C_R} f(z) dz.
\]

We consider how each piece behaves in the limit as \( R \to \infty \). The first summand is conveniently

\[
\int_{I_R} f(z) dz = \int_{0}^{R} f(x) dx \to \int_{0}^{\infty} \frac{1}{1 + x^n} dx \text{ as } R \to \infty.
\]

For the last summand, note that

\[
\left| \int_{C_R} f(z) dz \right| \leq \int_{C_R} \left| \frac{1}{1 + z^n} \right| dz \leq \int_{C_R} \frac{1}{R^n - 1} = \frac{2\pi R}{n} \frac{1}{R^n - 1} = \frac{4\pi R}{n(R^n - 1)} \to 0 \text{ as } R \to \infty.
\]

For the second summand, we can parameterize \( \omega^2 I_R \) by \( z = \omega^2 x \) for real \( x \in I_R \), with the proper orientation. In other words, we have

\[
\int_{\omega^2 I_R} f(z) dz = \int_{R}^{\infty} f(\omega^2 x) d(\omega^2 x) = \int_{R}^{\infty} \frac{1}{1 + (\omega^2 x)^n} \omega^2 dx = -\omega^2 \int_{0}^{\infty} \frac{1}{1 + \omega^2 x^n} dx = -\omega^2 \int_{I_R} f(z) dz.
\]

Thus, in the limit, we have

\[
-2\pi i \frac{\omega}{n} = \lim_{R \to \infty} \int_{\gamma} f(z) dz = \int_{0}^{\infty} \frac{1}{1 + x^n} dx - \omega^2 \int_{0}^{\infty} \frac{1}{1 + x^n} dx.
\]
So we have that
\[
\int_0^\infty \frac{1}{1 + x^n} dx = -2\pi i \frac{\omega}{n} (1 - \omega^2)
\]
\[
= \pi \frac{2ie^{\pi i/n}}{1 - e^{2\pi i/n}}
\]
\[
= \pi \frac{-2i}{e^{\pi i/n} - e^{-\pi i/n}}
\]
\[
= \pi \frac{2i}{e^{\pi i/n} - e^{-\pi i/n}}
\]
\[
= \pi \frac{1}{n \sin(\pi/n)}.
\]

\[\square\]

**Problem F13#3.** Suppose that \( f \) is analytic on \( \{ z : 0 < |z| < 1 \} \) and \( |f(z)| \leq \log \left( \frac{1}{|z|} \right) \). Show that \( f \) is identically 0.

**Solution.** First, we note that
\[
\lim_{z \to 0} |zf(z)| \leq \lim_{z \to 0} |z| \log(1/|z|) \leq \lim_{z \to 0} |z| \sqrt{1/|z|} = 0,
\]
so \( f \) has a removable singularity at 0. Thus there exists a function \( g \) that is analytic on \( D = \{ z : |z| < 1 \} \) and which satisfies \( g|_{D \setminus \{0\}} = f \). The condition \( |g(z)| \leq \log(1/|z|) \) means that \( |g(z)| \to 0 \) as \( |z| \to 1 \).

Let \( \overline{D}_r \) be a closed disc of radius 0 < \( r < 1 \). Since \( \overline{D}_r \) is compact and \( g \) is continuous, we know that for \( z \in \overline{D}_r \) we have \( |g(z)| \leq M_r \) for some real \( M_r \). But since \( |g(z)| \to 0 \) as \( r \to 1 \), we also have \( M_r \to 0 \). If \( R > r \), then clearly \( M_R \) is an upper bound on \( \overline{D}_r \) as well as on \( \overline{D}_R \) (because the latter contains the first). Thus \( |g(z)| \) is bounded by an arbitrarily small \( M_r \) on an arbitrarily large (still less than 1) disc.

\[\square\]

**Problem F13#6.** Determine all continuous functions on \( \{ z : 0 < |z| \leq 1 \} \) which are harmonic on \( \{ z : 0 < |z| < 1 \} \) and which are identically 0 on \( \{ z : |z| = 1 \} \).
Solution. First, note that the function $\log |z|$ has the desired properties, as does any real multiple. The general strategy here is to let $f(z) = \log |z|$ and let $g$ be a function with the given properties. Then we will show that they differ only by a multiplicative constant $c \in \mathbb{R}$ by modifying the Identity Principle for use with harmonic functions. We will also need the Möbius transformation $h(z) = \frac{z-i}{z+i}$ which maps the upper half plane $\mathbb{H}$ analytically to the unit disc, is continuous on the boundary ($\mathbb{R}$ to the unit circle), and sends $i \mapsto 0$.

Now both $f \circ h$ and $g \circ h$ are harmonic on $\mathbb{H} \setminus \{i\}$ and continuously map all of $\mathbb{R} \to \{0\}$. This is enough to apply Schwarz’s Reflection Principle and get two functions $F, G$ defined on $\mathbb{H} \setminus \{i\}$ which are harmonic everywhere away from $\mathbb{R}$ and still map $\mathbb{R} \to \{0\}$ continuously.

Since $F(z) = G(z) = 0$ for all $z \in \mathbb{R}$, we clearly have $\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 G}{\partial x^2} = 0$ on $\mathbb{R}$. Since both $F$ and $G$ are harmonic, we also have $\frac{\partial^2 F}{\partial y^2} = \frac{\partial^2 G}{\partial y^2} = 0$. Thus, along $\mathbb{R}$, we know that $\frac{\partial F}{\partial y}$ and $\frac{\partial G}{\partial y}$ are both real constants. Call them $\alpha$ and $\beta$ respectively. (They are real because both $F, G$ map into $\mathbb{R}$, so... duh.)

As with any harmonic function, we can now create the analytic functions

$$\frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y} \quad \text{and} \quad \frac{\partial G}{\partial x} - i\frac{\partial G}{\partial y}.$$ 

Note that their multiples

$$\beta \left( \frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y} \right) \quad \text{and} \quad \alpha \left( \frac{\partial G}{\partial x} - i\frac{\partial G}{\partial y} \right)$$

are equal on $\mathbb{R}$. (This is because, on $\mathbb{R}$, their first terms are 0 ($\frac{\partial F}{\partial x} = \frac{\partial G}{\partial x} = 0$), and their second terms are equal.) Thus, by the identity principle, we have

$$\beta \left( \frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y} \right) = \alpha \left( \frac{\partial G}{\partial x} - i\frac{\partial G}{\partial y} \right) \quad \text{on} \quad \mathbb{C} \setminus \{\pm i\}.$$ 

We integrate with respect to $x$ and get

$$\beta F - i\beta \int \frac{\partial F}{\partial y} \, dx = \alpha G - i\alpha \int \frac{\partial G}{\partial y} \, dx.$$ 

Considering only the real part of this equality we get $\beta F = \alpha G$. 

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Pushing each back through the inverse of our Möbius transformation gives us that

$$\frac{\beta}{\alpha} \log |z| = \frac{\beta}{\alpha} F \circ h^{-1} = G \circ h^{-1} = g(z)$$

for all $z \in \{z : 0 < |z| < 1\}$. \hfill \Box

\begin{center}
Spring 2013
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**Problem S13#3.** Prove that meromorphic functions on the extended complex plane are rational functions.

**Theorem (Liouville).** A bounded entire function must be constant.

**Solution.** Let $\mathbb{C}_\infty$ be the extended complex plane. Let $f: \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a non-constant meromorphic function, so all of it’s zeros and poles have finite order (by definition). It is easy to show that $f$ has a finite number of zeros and poles. Since $\mathbb{C}_\infty$ is compact, this is an immediate consequence of the identity principle for compact Riemann surfaces. OR Both $f(z)$ and $f(1/z)$ must have finitely many zeros and poles inside the closed unit disc $\overline{D}$ by the identity principle. Since $z \to 1/z$ is a bijective map between the zeros and poles of $f(z)$ outside $\overline{D}$ and the zeros and poles of $f(1/z)$ inside $\overline{D}$, we’re done.

Let $a_1, \ldots, a_k$ be the zeros of $f$ with corresponding orders $n_1, \ldots, n_k$. Let $b_1, \ldots, b_\ell$ be the poles of $f$ with corresponding orders $m_1, \ldots, m_\ell$. Let

$$g(z) = \frac{\prod_{i=1}^k (z - a_i)^{n_i}}{\prod_{j=1}^\ell (z - b_j)^{m_j}}.$$

Note that $g$ is a meromorphic function on $\mathbb{C}_\infty$ with zeros and poles at precisely the same points as $f$. Let

$$h(z) = \frac{f(z)}{g(z)}.$$

Since $f$ and $g$ have the same poles, $h$ is a meromorphic function with no zeros and no poles and extends to a nonzero, bounded, entire function. By Liouville, $h(z) = c \in \mathbb{C}$ must be constant. Thus $f(z) = cg(z)$ is a rational function. \hfill \Box
Problem S13#4. Suppose $u$ is harmonic and bounded in $\{ z \in \mathbb{C} : 0 < |z| < 1 \}$. Show that 
$\{ z = 0 \}$ is a removable singularity of $u$. That is, show that $u(0)$ can be defined so that $u$ becomes harmonic in the full disk $\{|z| < 1\}$.

Theorem (Schwarz’s). Given a piecewise continuous function $u_r(z)$ defined on the circle of radius $r$, the Poisson Integral

$$P_{u_r}(z) = \frac{1}{2\pi} \int_0^{2\pi} \text{Re} \left( \frac{re^{i\theta} + z}{re^{i\theta} - z} \right) u_r(re^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} u_r(re^{i\theta}) d\theta$$

defines a harmonic function inside the disc of radius $r$. Furthermore, $\lim_{z \to re^{i\theta}} P_{u_r}(z) = u_r(re^{i\theta})$, so that $P_{u_r} = u_r$ on the circle of radius $r$.

Solution. Our approach is to show that $u$ is given by the Poisson Integral as it appears in Schwarz’s Theorem. Let $D_r$ be the open disc of radius $r$ and let $C_r$ be the circle of radius $r$, both centered at the origin. Let $D$ and $C$ simply be the unit disc and circle respectively. Fix an arbitrary $r \in (0, 1)$, and define a function $g$ on $\{ z : 0 < |z| \leq r \}$ by

$$g(z) = u(z) - P_{u_r}(z) \quad \text{where} \quad u_r = u|_{C_r}.$$ 

Note that on $|z| = r$, we know that $P_{u_r} = u_r = u$, so that $g(z) = 0$. Furthermore, we know that $g$ is bounded because $P_{u_r}$ is continuous on the closed set $D_r \cup C_r$ (and thus bounded), and $u$ is bounded by assumption. Now, for any $\varepsilon > 0$, define

$$g_\varepsilon(z) = g(z) + \varepsilon \log(\frac{|z|}{r}).$$

Note that $g_\varepsilon(z) = g(z) = 0$ on $|z| = r$.

Since $g$ is bounded, we have

$$\limsup_{z \to 0} g_\varepsilon(z) < 0.$$ 

Thus, there is some $\delta > 0$ such that $g_\varepsilon(z) \leq 0$ for all $|z| \leq \delta$. Now we have that $g_\varepsilon(z) \leq 0$ on $C_\delta$ and $g_\varepsilon(z) = 0$ on $C_r$. By applying the maximum modulus principle for harmonic functions to $g_\varepsilon$
on the closed annulus bounded by $C_\delta$ and $C_r$, we know that $g_\delta(z) \leq 0$ for all $\delta < |z| < r$. Taking the limit $\varepsilon \to 0$ gives us that $g(z) \leq 0$ for all $\delta < |z| < r$. But since this argument applies for any positive $\delta' < \delta$, we have really shown that

$$g(z) \leq 0 \quad \text{on} \quad D_r \setminus \{0\}.$$ 

By making an analogous argument for the function $-g$, we can concluded that $g(z) = 0$ on its entire domain $\{z : 0 < |z| \leq r\}$.

Since $u = P_{u_r}$ (in $D_r$) for any $r \in (0,1)$, we can define $u(0) = P_{u_r}(0)$ to make $u$ harmonic in all of $D$. \square

**Problem S13#7.** Find a one to one holomorphic map that sends the unit disc $D(0,1)$ onto the slit disc $D(0,1) \setminus \{(0,1)\}$.

**Solution.** We'll freely use the Möbius transformation $h: \mathbb{H} \to D$ defined by

$$h(z) = \frac{z - i}{z + i} = \frac{x + i(y - 1)}{x + i(y + 1)} = \frac{x^2 + y^2 - 1}{x^2 + (y + 1)^2} + i \frac{-2x}{x^2 + (y + 1)^2}$$

and the fact that it is invertible and analytic. Furthermore, note, from the real and imaginary parts, that $h$ send the upper right quadrant to the lower half disc. Choose an appropriate branch cut (say at the negative real axis) such that $\sqrt{z}$ is a bijective, analytic function from the upper half plane to the first quadrant. Letting $Q_1$ be the first quadrant and $D^-$ be the open lower half disc, we now have

$$\left(h(\sqrt{h^{-1}(z)})\right)^2: D \to \mathbb{H} \to Q_1 \to D^- \to D \setminus (0,1)$$

as desired. \square

**Problem S13#8.** Show that the total number of poles of an elliptic function $f$ in its fundamental parallelogram is $\geq 2$.

To clarify, the problem is false as stated. We must assume that all elliptic functions are non constant, and also that we are counting poles with multiplicity.
**Solution.** Let $f$ be an elliptic function with period lattice $w_1\mathbb{Z} + w_2\mathbb{Z}$ having fundamental parallelogram $P$ based at a point $a$ such that $\partial P$ has no poles on it. First, note that $\mathcal{P}$ is compact and thus $f$ is bounded. If $f$ has no poles in $P$, then $f$ is constant by Liouville’s Theorem.

Now suppose, for the sake of contradiction, that $f$ has a single simple pole in $P$. Then by the Residue’s Theorem,

$$\int_{\partial P} f(z) \, dz \neq 0.$$ 

On the other hand,

$$\int_{\partial P} f(z) \, dz = \int_{a}^{a+w_1} f(z) \, dz + \int_{a+w_1}^{a+w_1+w_2} f(z) \, dz + \int_{a+w_1+w_2}^{a} f(z) \, dz + \int_{a}^{a+w_2} f(z) \, dz$$

$$= \int_{a}^{a+w_1} f(z) \, dz - \int_{a+w_2}^{a+w_1+w_2} f(z) \, dz + \int_{a+w_1}^{a+w_1+w_2} f(z) \, dz - \int_{a}^{a+w_2} f(z) \, dz.$$ 

And since, by the periodicity of $f$, we have

$$\int_{a}^{a+w_1} f(z) \, dz = \int_{a+w_1+w_2}^{a+w_1} f(z) \, dz \quad \text{and} \quad \int_{a+w_2}^{a+w_1+w_2} f(z) \, dz = \int_{a}^{a+w_2} f(z) \, dz,$$

we find that $\int_{\partial P} f(z) \, dz = 0$, which contradicts the Residue Theorem above. 

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**Fall 2012**

**Problem F12#1.** Show that the Möbius transformation maps a straight line or circle onto a straight line or circle.

**Solution.** We will freely use two basic facts. First, the group of Möbius transformations is generated by translations, complex-scalar dilations (i.e. including rotation), and the inversion $f : z \to \frac{1}{z}$. Second, the set of all circles and lines in the complex plane is equal to the set of solutions $z = x + iy$ to equations of the form

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

where $A, B, C, D \in \mathbb{C}$ and where $B^2 + C^2 > 4AD$ (non-degenerate). In particular, the solution set is a line if $A = 0$ and a circle otherwise. It is clear that a translated line is still a line, that a dilated
circle is still a circle, etc. so all we need to show is that a solution set to an equation of the form above is unchanged by the inversion \(f\).

Fix some arbitrary \(A, B, C, D \in \mathbb{C}\) such that \(B^2 + C^2 > 4AD\). Let \(S = \{z = x + iy : A(x^2 + y^2) + Bx + Cy + D = 0\}\). Note that in terms of real and imaginary parts, we have
\[
f(x + iy) = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} = u + iv.
\]
Furthermore, since \(f = f^{-1}\), we also have \(x = \frac{u}{u^2 + v^2}\) and \(y = \frac{-v}{u^2 + v^2}\). Thus the image \(f(S)\) is the set of all points \(1/z = u + iv\) that solve
\[
A\left(\frac{u}{u^2 + v^2}\right)^2 + B\left(\frac{u}{u^2 + v^2}\right) + C\left(\frac{-v}{u^2 + v^2}\right) + D = 0.
\]
Clearing the denominator gives us
\[
A + Bu - Cv + D(u^2 + v^2) = 0
\]
which is another non-degenerate quadratic equation of the same form because we still have \(B^2 + (-C)^2 > 4AD\). Thus, the image of \(S\) under the inversion \(f\) is still a circle or a line.

\(\square\)

**Problem F12#2.** Let \(f\) be a complex valued function in the unit disc \(D(0,1)\) such that \(g = f^2\) and \(h = f^3\) are both analytic. Prove that \(f\) is analytic in \(D(0,1)\).

**Solution.** Since \(g, h\) are both analytic, we know that \(f = \frac{h(z)}{g(z)}\) is meromorphic. Supposing that \(f\) is not analytic, \(f\) must have a pole. Let \(a \in D\) be an order \(n\) pole of \(f\). Then \(f(z) = \frac{r(z)}{(z-a)^n}\) for some function \(r(z)\) which is analytic and nonzero near \(a\). Clearly \(r^2\) is also analytic and nonzero around \(a\), so \(g(z) = (f(z))^2 = \frac{r(z)^2}{(z-a)^{2n}}\) has a pole of order \(2n\) at \(a\). This contradicts the analyticity of \(g\). So \(f\) has no poles and is analytic on \(D\).

\(\square\)

**Problem F12#6.** Let \(f(z)\) be analytic on the upper half-plane \(\{z \in \mathbb{C} : \text{Im}(z) > 0\}\). Suppose that \(|f(z)| < 1\) for all \(z\) in the domain of \(f\), and \(f(i) = 0\). Find the largest possible value of \(|f(2i)|\).

**Lemma (Schwarz).** Let \(f : D \to D\) be an analytic function with \(f(0) = 0\). Then

\(\square\)
• $|f(z)| \leq |z|$ for all $z \in D$ and $|f'(0)| \leq 1$

• If $|f(z)| = |z|$ for some $z \neq 0$ or if $|f'(0)| = 1$, then $f(z) = cz$ for some constant with $|c| = 1$.

Solution. Let $\mathbb{H}$ be the upper half plane, let $D$ be the unit disc, and let $f$ be as in the problem statement. We freely use the Möbius transformation $h: \mathbb{H} \to D$ defined by $h(z) = \frac{z-i}{z+i}$ and the facts that $h(i) = 0$ and that $h$ is invertible. Now the function $F = f \circ h^{-1}: D \to D$ and has $F(0) = 0$. Thus, we can apply Schwarz Lemma to $F$. We compute $h(2i) = \frac{2i-i}{2i+i} = 1/3$. Thus

$$|f(2i)| = |F \circ h(2i)| = |F(1/3)| \leq 1/3.$$

Since $h$ satisfies all the desired properties, it is indeed possible to have a function with $|f(2i)| = 1/3$. \qed

Problem F12#7. Suppose $f(z)$ be analytic on the punctured unit disk $D(0,1) \setminus \{0\}$, and the real part of $f(z)$ is positive. Prove that $f$ has a removable singularity at 0.

Solution. Let $\mathbb{H} = \{ z : \text{Re} z > 0 \}$. We freely use the invertible, conformal map $h: \mathbb{H} \to D$ defined by $h(z) = \frac{z-1}{z+1}$. Note that $h(1) = 0$. Now $h \circ f: D \setminus \{0\} \to D \setminus \{0\}$ is analytic and bounded. Thus

$$\lim_{z \to 0} |z \cdot h \circ f(z)| = 0$$

and $h \circ f$ has a removable singularity at 0 and thus an analytic continuation $g$ to the whole disc $D$. Since $g|_{D \setminus \{0\}} = h \circ f$ is continuous and since 0 is a limit point of $D \setminus \{0\}$, we know that $g(0)$ must also be a limit point of $D \setminus \{0\}$, i.e. $g(0) \in \overline{D \setminus \{0\}} = \overline{D}$. By the Open Mapping Theorem, we have either that $g(D)$ is open, and must then be a subset of $D$, or else that $g$ is constant on $D$, in which case $g(D) = a \in D \setminus \{0\}$. Either way, we now have $g: D \to D$ so we can take $h^{-1} \circ g$ and have that $h^{-1} \circ g|_{D \setminus \{0\}} = h^{-1} \circ h \circ f$. Thus we have created an analytic continuation of $f$ at 0, and so the singularity was removable. \qed

Spring 2012
Problem S12#2.  

a. State Rouché’s Theorem.

b. Use Rouché’s Theorem to find the number of zeros of the polynomial \(z^4 + 5z + 3\) in the annulus \(1 < |z| < 2\).

Solution.  

a. 

**Theorem (Rouché’s).** Let \(f, g\) be analytic functions on some open, connected set \(\Omega\). Let \(\gamma \subset \Omega\) be a closed simple curve with winding number \(n(\gamma, z) = 0\) or \(1\) for all \(z \in \Omega\). If

\[
|f(z) - g(z)| < |g(z)| \quad \text{for all} \quad z \in \gamma,
\]

then \(f\) and \(g\) have the same number of zeros on the interior of \(\gamma\) (with multiplicity).

(Note: because of the absolute value, the roles of \(f\) and \(g\) are interchangeable.)

b. This is a straightforward application of the theorem to the circles \(\gamma_1 = \{z : |z| = 1\}\) and \(\gamma_2 = \{z : |z| = 2\}\). Note that there are at most four zeros total.

First, let \(f(z) = z^4 + 5z + 3\) and \(g(z) = 5z\). Now

\[
|f(z) - g(z)| = |z^4 + 3| \leq |z^4| + |3| = 4 < 5 = |g(z)| \quad \text{on} \quad \gamma_1.
\]

Thus \(f\) has one zero inside the unit circle.

Second, let \(f(z) = z^4 + 5z + 3\) and \(g(z) = z^4\). Now

\[
|f(z) - g(z)| = |5z + 3| \leq |5z| + |3| = 13 < 16 = |g(z)| \quad \text{on} \quad \gamma.
\]

Thus \(f\) has four zeros inside the circle of radius two.

We must now check the boundaries. In this case, all the zeros are accounted for within \(\gamma_2\), so we need only check that there are no zeros on the unit circle. Note that by the reverse triangle inequality, we have

\[
|f(z)| \geq |5z| - |z^4| - 3 = 5 - 3 - 1 = 1 \quad \text{on} \quad \gamma_1.
\]
So \( f(z) \neq 0 \) on \( \gamma_1 \).

Thus we know that there are three zeros in the annulus \( 1 < |z| < 2 \).

\[ \square \]

**Problem S12#4.** Prove that there is no function \( f \) that is analytic on the punctured disk \( \{ z \in \mathbb{C} : 0 < |z| < 1 \} \), and \( f' \) has a simple pole at 0.

**Solution.** Suppose for the sake of contradiction, that such an \( f \) does exists. Then in a punctured neighborhood \( V \setminus \{ 0 \} \) of 0, we know that

\[
f'(z) = \frac{c}{z} + g(z)
\]

where \( c \in \mathbb{C} \) is nonzero and \( g(z) \) is analytic on \( V \). But then there exists an antiderivative \( G \) of \( g \) on \( V \). But now we have \( 1/z = \frac{1}{c}(f'(z) - g(z)) \) and can take the antiderivative in \( V \setminus \{ 0 \} \) to find that \( \frac{1}{c}(f - G) \) is an antiderivative of \( 1/z \) and is analytic in \( V \setminus \{ 0 \} \). However, since \( \int_\gamma \frac{1}{z} \, dz \neq 0 \) for a closed loop \( \gamma \subset V \setminus \{ 0 \} \) around zero, we know (by a precursor to Cauchy’s Theorem) that \( 1/z \) cannot have an analytic antiderivative.

\[ \square \]

**Problem S12#8.** State and prove a version of the Schwarz Reflection Principle.

**Solution.**

**Theorem (Schwarz Reflection Principle).** Let \( \Omega = \Omega^+ \cup \Omega^- \cup I \) be a region symmetric with respect to the real axis (with the components you’d expect...). Let \( u(x,y) \) be harmonic on \( \Omega^+ \), continuous on \( I \cup \Omega^+ \), and identically zero on \( I \). Then

\[
U(x,y) = \begin{cases} 
  u(x,y) & \text{if } y > 0 \\
  -u(x,-y) & \text{if } y < 0 
\end{cases}
\]

is a harmonic extinction of \( u \) on all of \( \Omega \).

**Proof.** ...
Problem S11#8. b) Use Rouche’s Theorem to prove the Fundamental Theorem of Algebra.

Theorem (Fundamental Theorem of Algebra). Every non-constant polynomial with coefficients in $\mathbb{C}$ has at least one root in $\mathbb{C}$.

Solution. Let $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a non-constant polynomial in $\mathbb{C}[z]$. Choose $R$ such that $R > 1$ and

$$R > |a_{n-1}| + |a_{n-2}| + \cdots |a_0|.$$ 

Let $g(z) = z^n$. Clearly, both $f, g$ are analytic on the disc $|z| < R + 1$. Let $\gamma$ be the circle $|z| = R$. Now

$$|f(z) - g(z)| = |a_{n-1}z^{n-1} + \cdots + a_0| \leq |a_{n-1}||z|^{n-1} + |a_{n-2}||z|^{n-2} + \cdots |a_0|$$

$$= |a_{n-1}|R^{n-1} + |a_{n-2}|R^{n-2} + \cdots |a_0|$$

$$\leq R^{n-1}(|a_{n-1}| + |a_{n-2}| + \cdots |a_0|)$$

$$< R^n = |g(z)|.$$ 

So we know that $f$ has $n$ zeros inside $\gamma$. \hfill \Box

Problem Madeitup. Use Liouville’s Theorem to prove the Fundamental Theorem of Algebra.

Solution. Let $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a non-constant polynomial in $\mathbb{C}[z]$. Suppose for the sake of contradiction that $f(z)$ has no zeros. Then the function $1/f(z)$ has no poles, and is thus entire. We wish to show that $1/|f(z)|$ is bounded. We can write

$$\frac{1}{|f(z)|} = \frac{1}{|z|^n} \left| 1 + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n} \right|$$

and see that as $z \to \infty$, each term in the denominator $|\frac{a_{n-i}}{z^i}| \to 0$. Thus $1/|f(z)| \to \frac{1}{|z|^n} \to 0$ as $z \to \infty$. So we can choose an $R$ large enough such that for all $|z| > R$, we have that $1/|f(z)| < 1$.

But then we have that the continuous function $1/|f(z)|$ must be bounded on the compact set
\(|z| \leq R\) by the Maximum Modulus Principle. Thus \(1/|f(z)|\) is in fact bounded everywhere in \(\mathbb{C}\). By Liouville, \(1/|f(z)|\) is constant, but this contradicts the fact that \(f(z)\) is non-constant. Thus \(f\) has a 0. \(\square\)