Exercise 0.1 (Spring 2010, Midterm 2).
Evaluate \( \int_C xy \, dx + x^2 \, dy \) where \( C \) is the boundary of the triangle with vertices \((0, 0), (2, 0), (1, 1)\) oriented in the counter clockwise direction.

We want to use Green's Theorem.

So we write

\[
\int_C F \cdot ds = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dA
\]

So \( F(x, y) = (xy, x^2) \) and \( \frac{\partial F_1}{\partial y} = 2x \) while \( \frac{\partial F_2}{\partial x} = y \).

Thus,

\[
\int_C xy \, dx + x^2 \, dy = \iint_D 2x - y \, dA
\]

\[
= \iint_D x \, dA + \int_0^2 \int_0^x x \, dy \, dx
\]

\[
= \frac{1}{2}
\]

Exercise 0.2.

Use a line integral to compute the area inside the ellipse \( \frac{x^2}{16} + \frac{y^2}{9} = 1 \).

**Bonus:** Is there another force field \( \mathbf{F} \) that makes the integration easier?

We know Area = \( \iint_D 1 \, dA \)

We want to use Green's Theorem and write \( \iint_D 1 \, dA = \iint_D F \cdot ds \), so we need to find the right \( F \) and \( \mathbf{F} \).

\( F \): We just need \( F \) to satisfy \( L = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1 \).

So \( F(x, y) = (\frac{3}{2}, \frac{1}{2}) \) will do (check this on your own).

\( C \): This should be a parametrization of the ellipse.

Start w/ param. for circle: \( (\cos(t), \sin(t)) \) with \( 0 \leq t \leq 2\pi \).

Multiply each component to "stretch/squish" the shape.

\( C(t) = (4 \cos(t), 3 \sin(t)) \rightarrow C'(t) = (-4\sin(t), 3 \cos(t)) \)

Now we can compute

\[
\int_C F \cdot ds = \int_0^{2\pi} F(4 \cos(t), 3 \sin(t)) \cdot (-4\sin(t), 3 \cos(t)) \, dt
\]

\[
= \int_0^{2\pi} 12 \sin^2(t) + 6 \cos^2(t) \, dt
\]

\[
= 6\int_0^{2\pi} 1 \, dt = 6 \cdot 2\pi = 12\pi
\]
Exercise 0.3.
Let \( C_1 \) be the counter-clockwise circle of radius 3 centered at the origin and let \( C_2 \) be the clockwise circle with radius \( \frac{1}{2} \) centered at the point \((1,1)\). Letting \( \mathbf{F}(x,y) = (-y,0) \), compute the line integral \( \int_{C_1 + C_2} \mathbf{F} \cdot ds \).

First, we break up the line integral
\[
\int_{C_1 + C_2} \mathbf{F} \cdot ds = \int_{C_1} \mathbf{F} \cdot ds + \int_{C_2} \mathbf{F} \cdot ds = \int_{C_1} \mathbf{F} \cdot ds - \int_{C_2} \mathbf{F} \cdot ds
\]
changing to counter-clockwise orientation introduces a negative.

New notice that \( \frac{dy}{dx} = \frac{-2y}{2x} = \frac{-y}{x} = 0 \) (\(-\)) = 1

So applying Green's Theorem to the right-hand side we got
\[
\iint_{C_1} 1 \, dA - \iint_{C_2} 1 \, dA = \text{where the integration is now over the inside of } C_1 \text{ and } C_2
\]

These two integrals are simply the area inside each circle \( C_1 \) and \( C_2 \)

Using \( A = \pi r^2 \) we calculate \( \int_{C_1} \mathbf{F} \cdot ds \) as \( \int_{C_1} 1 \, dA - \int_{C_2} 1 \, dA = 4\pi - \frac{1}{4} \pi = \frac{15}{4} \pi \)

Exercise 0.4 (Spring 2007, Midterm 2).
Let \( W \) be the region in \( \mathbb{R}^3 \) inside the cylinder \( z^2 + y^2 = 1 \) and bounded by the \( yz \)-plane and the plane \( z + x = 1 \). If \( f(x,y,z) \) is any scalar function, set up an integral for \( f \) over the region \( W \) that ends with

- \( \int dx \, dz \, dy \)
- \( \int dy \, dz \, dx \)

The solution posted online has a really good explanation complete with MATHEMATICA pictures!

**Bonus:** If \( f(x,y,z) = 1 \) for all \( x,y,z \) what do the two integrals represent?

**Bonus Bonus:** Can you compute the volume of \( W \)? What's the easiest way?

**Bonus Bonus Bonus:** Can you set up a double integral for the volume?

- Using the shadow method from Math Insights...

  \[
  \iint_{\text{shadow}} f(x,y,z) \, dx \, dA
  \]

  This shadow is the circle \( z^2 + y^2 = 1 \) so
  \[
  \iint_{\frac{1}{4} - y^2}^{1} f(x,y,1-x) \, dx \, dy
  \]

  This time, the "shadow" or cross-section is in the \( zy \)-plane and is the triangle

- \( z = 1 - x \)

  \[
  \iint_{\text{shadow}} f(x,y,z) \, dA = \int_{0}^{1} \left( \int_{1-x}^{1} f(x,y,1-x) \, dx \right) dy
  \]

  Blue + red together is the intersection of the plane + the cylinder.