MAHONIAN Z STATISTICS

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ABSTRACT. The Z statistic of Zeilberger and Bressoud is computed by sum-
ing the major index of the 2-letter subwords. We generalize this idea to other splittable Mahonian statistics. We call splittable Mahonian statistics which produce other splittable Mahonian statistics in this fashion Z-Mahonian. We characterize Z-Mahonian statistics and include several examples.

1. INTRODUCTION

The major index statistic on words was first studied by Major P.A. MacMahon in the early part of the 20th century [16]. As MacMahon showed, the distribution of the major index on all rearrangements of a given word is the same as that of the inversion number. The significance of MacMahon’s work has been recognized by giving the name Mahonian to any statistic whose distribution is the same as that of the inversion number (or the major index). In recent years, several other Mahonian statistics on words have emerged. These include the interpolating statistics of Rawlings [17], Kadell [13] and White [22], and an important new statistic due to Denert [7]. Also, Simion and Stanton [20] introduced new Mahonian statistics on binary words.

Of special interest is the Z statistic introduced by Zeilberger and Bressoud [23], obtained by summing the major index for each of the 2-letter subwords of a given word. Zeilberger and Bressoud showed that the Z statistic was Mahonian, a key step in their proof of the q-Dyson conjecture.

At this juncture, we emphasize that the Z statistic and the statistic from which it was manufactured have the same distribution. It is, therefore, natural to look for other statistics which have this “Z-ing” property. We restrict our attention to statistics on binary words which are splittable, an idea introduced in earlier work [9], [10]. In this paper we give precise conditions under which a family of splittable Mahonian statistics on binary words can be “Z-ed” to give a splittable Mahonian statistic on words. We call statistics with this property Z-Mahonian.

The next section defines words and statistics on words, and includes a more extended discussion of Mahonian statistics. In Section 3 we discuss the notion of a splittable Mahonian statistic and restate a key result from [10]. The definition of the Z operator that extends the ideas of Zeilberger and Bressoud and our main theorems appear in Section 4; illustrative examples are provided in Section 5.

In Section 6 we return to the topic of splittable statistics with several counting theorems. The paper concludes with a few remarks about the general applicability of our Z-operator results.

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2. Mahonian statistics

In this section we first define words and statistics on words. We will introduce the important class of statistics on words, Mahonian statistics.

Let $K$ be a positive integer. A letter is an element of $[K] = \{1, 2, \ldots, K\}$. A word is a sequence of letters. If $w$ is a word, then the $i$-th letter is $w_i$. The length of $w$ is the number of letters in $w$. A letter $x$ appears in $w$ if $w_i = x$ for some $i$. If $x$ and $y$ are letters, then $w[x, y]$ is the subword obtained by removing all letters except $x$ and $y$ from $w$.

The type of $w$, $\rho = (\rho_1, \rho_2, \ldots, \rho_K)$, is a vector representing the number of each kind of letter in $w$. That is, $\rho_1$ is the number of 1’s in $w$. Thus, if the type of $w$ is $\rho$, then $x$ appears in $w$ if and only if $\rho_x > 0$. Let $a(\rho)$ denote the number of letters appearing in words of type $\rho$ and let $|\rho|$ denote the length of any word of type $\rho$.

The set of all words of type $\rho$ is denoted $W_\rho$. If $x$ appears in words of type $\rho$, we write $x \in \rho$. The new type $\rho - x$ is $(\rho_1, \ldots, \rho_i - 1, \ldots, \rho_K)$. That is, one $x$ has been removed from the type.

An alternative description of type is an exponential form. If $w$ has type $\rho$, then we write $1^{\rho_1}2^{\rho_2}\ldots K^{\rho_K}$ to represent the type. This notation is useful because if $x$ does not appear in $w$, then the term $x^0$ won’t appear in the exponential form.

If $\rho$ and $\mu$ are two types, then $\mu < \rho$ if $\mu_i \leq \rho_i$ for all $1 \leq i \leq K$, with a strict inequality at least once. If $\mu < \rho$, define $\rho - \mu = (\rho_1 - \mu_1, \ldots, \rho_K - \mu_K)$.

Two special kinds of types are important. First, binary words are words with only two letters appearing. Second, permutations are words such that each letter which appears appears only once.

A statistic on words in $W_\rho$ is a function $s$ from $W_\rho$ to the non-negative integers. If the type is not specified, then the function is from all words to the non-negative integers. If $s$ is a statistic on words of type $\rho$, then the corresponding statistic generating function is the polynomial

$$\sum_{w \in W_\rho} q^{s(w)}.$$

A statistic $s$ on words of type $\rho$ is Mahonian if its statistic generating function is

$$\left[\frac{|\rho|}{\rho}\right]_q \equiv \left[\frac{|\rho|}{\rho_1, \rho_2, \ldots, \rho_K}ight]_q,$$

called the $q$-multinomial coefficient, and defined by

$$\left[ \begin{array}{c} n \\ n_1, n_2, \ldots, n_k \end{array} \right]_q = \frac{[n]_q!}{[n_1]_q! [n_2]_q! \cdots [n_k]_q!},$$

$$[n]_q! = [n]_q [n - 1]_q \cdots [1]_q,$$

and

$$[m]_q = 1 + q + q^2 + \cdots + q^{m-1}.$$ 

In the case of binary words, the $q$-multinomial coefficient becomes the $q$-binomial coefficient. In the case of permutations, the $q$-multinomial coefficient becomes the $q$-analogue of $n!$.

Mahonian statistics on words have a long and important history. The interested reader is referred to [10] for a summary of this history. Over the years, many Mahonian statistics have been discovered and rediscovered. We list here some of the important ones, and give some of their definitions.
Define\[\text{INV}(w) = \# \{(i, j) \mid i < j, w_i > w_j\}\]
and\[\text{MAJ}(w) = \sum_{\{i \mid w_i > w_{i+1}\}} i.\]

The statistics \text{INV} and \text{MAJ} are classical Mahonian statistics, dating back to MacMahon [16].

In their proof of the \textit{q}-Dyson conjecture, Zeilberger and Bressoud were led to another Mahonian statistic, which we call \text{ZLB}. (In the literature this statistic is referred to as \textit{Z}.) This statistic interpolates between \text{INV} on permutations and \text{MAJ} on binary words. It is defined by
\[\text{ZLB}(w) = \sum_{x \leq y} \text{MAJ}(w[x, y]).\]

In a later paper [3], Bressoud describes “tournamented” statistics. His idea is extended in Section 4. Other Mahonian interpolating statistics include \text{RAW} [17], \text{KAD} [13] and \text{WHT} [22].

Another important Mahonian statistic first appeared in the work of Denert [7]. This statistic, \text{DEN}, was first proved Mahonian on permutations by Foata and Zeilberger [8], then on words by Han [11].

All the above statistics are defined on all words. However, many Mahonian statistics are defined only on permutations. These include some recent statistics due to Babson and Steingrimsson [2].

Since our plan is to bootstrap a Mahonian statistic on binary words up to all words, our discussion will not include Mahonian permutation statistics.

3. Splittable statistics

In this section, we introduce the idea of a splittable Mahonian statistic, and we review a couple of the results in [10].

The idea of a splittable statistic is described in detail in [10]. Roughly speaking, a statistic on words is splittable if it can be written as a sum of two pieces. One piece corresponds to the rightmost letter. The second piece corresponds to the subword obtained by removing the rightmost letter. The second piece must be (recursively) splittable.

More precisely, a Mahonian statistic \text{s} on \(W_\rho\) is splittable (or splits) if, for \(v = wx \in W_\rho\),
\[s(wx) = \alpha_x + T(w),\]
where \(T\) is a Mahonian statistic on \(W_{\rho - x}\) which splits, and \(\alpha_x\) is an integer.

The following Theorem is a consequence of Theorem 6.2 in [10].

**Theorem 1.** A Mahonian statistic \(s\) is splittable on words of type \(\rho\) if and only if for every \(v \in W_\mu, 0 \leq \mu < \rho\), there is a permutation of the letters in \(\rho - \mu\), \(\pi_v\), such that if \(w = x_1x_2 \ldots x_n \in W_\rho\) with \(u_i = x_1 \ldots x_i\) and \(v_i = x_{i+1} \ldots x_n\),
then
\[ s = \sum_{i=1}^{n} \alpha_i, \]
where
\[ \alpha_i = \# \{ y \in u_i \mid \pi_v y > \pi_v x_i \}. \]

For example, if \( \rho = (1, 2, 1) \), then \( s \) defined below is splittable Mahonian.

\[
\begin{align*}
    s(1223) &= 3 \\
    s(1232) &= 1 \\
    s(3122) &= 1 \\
    s(2231) &= 5 \\
    s(2123) &= 2 \\
    s(2132) &= 2 \\
    s(2312) &= 3 \\
    s(2321) &= 3 \\
    s(2213) &= 4 \\
    s(1322) &= 0 \\
    s(3212) &= 2 \\
    s(3221) &= 4
\end{align*}
\]

Thus, \( s(2312) = 3 \), since \( \pi_0 = (123) \), \( \pi_2 = (1)(23) \), \( \pi_{12} = (23) \), and \( \pi_{312} = (2) \), so that \( \alpha_4 = 0 \), \( \alpha_3 = 2 \), \( \alpha_2 = 1 \) and \( \alpha_1 = 0 \).

In what follows, we will identify the permutation \( \pi_v \) with the corresponding transitive tournament on the letters in the permutation.

Many Mahonian statistics on words appearing in the literature are splittable. These include \( \text{maj} \), \( \text{inv} \), \( \text{den} \), \( \text{raw} \), \( \text{kad} \), \( \text{wht} \), and \( \text{zlb} \). These examples all appear in [10].

The fact that \( \text{maj} \) splits requires special attention. It splits because of an “encoding” due to Han [11] which gives exactly Equation (1). This encoding is as follows. For \( x = w_j \) in the word \( w \), \( \alpha_j \) counts the number of letters to the left of position \( j \) and cyclically between \( x \) and \( y = w_{j+1} \) (with \( w_{n+1} = k \)), counting \( y \)'s but not \( x \)'s

For example, if 
\[ w = 21143422313, \]
then the vector of \( \alpha_j \)'s is 
\[ (0, 0, 1, 3, 1, 3, 0, 1, 4, 5, 2). \]
Thus, \( \text{maj}(w) = 20 \).

4. The \( Z \) Operator

In this section we introduce the \( Z \) operator and we give two necessary and sufficient conditions for the \( Z \) operator to produce a splittable Mahonian statistic.

We now consider words of type \( \rho \) which use \( K \) letters. We call a collection of statistics, one for each type \( x^p y^p z^p \), a binary family of type \( \rho \). If each statistic in a binary family of type \( \rho \) is splittable Mahonian, we say the binary family is splittable Mahonian. Suppose \( s \) is a binary family of type \( \rho \). If \( w \) is a word of type \( x^p y^p z^p \), we abuse notation and write \( s(w) \) to denote the value of the statistic in the collection \( s \) on \( w \).

Now suppose \( s \) is a binary family of type \( \rho \). Define \( Z(s) \) on words of type \( \rho \) as follows:
\[ Z(s) = \sum_{\{x,y\}} s(w[x, y]). \]

It is obvious that if \( s = \text{inv} \), then \( Z(s) = \text{inv} \). In [23] it was shown that if \( s = \text{maj} \), then \( Z(s) \) is Mahonian. This is the statistic \( \text{zlb} \) described in Section 2. A combinatorial proof that it is Mahonian was given in [12], and a proof that it is splittable appears in [10]. The statistic \( \text{zlb} \) interpolates between \( \text{inv} \) and \( \text{maj} \) in the following sense. On binary words we have \( \text{zlb} = \text{maj} \). On permutations we have \( \text{zlb} = \text{inv} \).
The goal of the rest of this paper is to give general criteria under which $Z(s)$ is a splittable Mahonian statistic. We call binary families with this property $Z$-Mahonian.

The following example will serve as illustration. Define EXS on 4 letters as follows. Let \( \text{EXS} = \text{INV} \) on words using 1's and 2's, on words using 1's and 3's, and on words using 1's and 4's. Also, let \( \text{EXS} = \text{INV} \) on words using 2's and 3's. Finally, on words using 2's and 3's or words using 3's and 4's, let \( \text{EXS} = \text{MAJ} \). The following table describes EXS:

$$
\begin{array}{ccc}
2 & 3 & 4 \\
\text{INV} & \text{INV} & \text{INV} \\
\text{MAJ} & \text{INV} & \\
\text{MAJ} & \\
\end{array}
$$

Then for any type $\rho$, EXS is a binary family of splittable Mahonian statistics of type $\rho$. For instance, if

$$w = 21143422313,$$

then $Z(\text{EXS})(w) = 19$, since

$$
\begin{align*}
\text{EXS}(w[1, 2]) &= \text{INV}(211221) = 5 \\
\text{EXS}(w[1, 3]) &= \text{INV}(113313) = 2 \\
\text{EXS}(w[1, 4]) &= \text{INV}(11441) = 2 \\
\text{EXS}(w[2, 3]) &= \text{MAJ}(232233) = 2 \\
\text{EXS}(w[2, 4]) &= \text{INV}(24422) = 4 \\
\text{EXS}(w[3, 4]) &= \text{MAJ}(43433) = 4
\end{align*}
$$

For another example, take EXT to be the binary family given by this table:

$$
\begin{array}{ccc}
2 & 3 & 4 \\
\text{INV} & \text{INV} & \\
\text{MAJ} & \text{INV} & \text{INV} \\
\text{MAJ} & \text{INV} & \text{INV} \\
\end{array}
$$

Again, EXT is a binary family of splittable Mahonian statistics of type $\rho$. If, for example,

$$w = 21143422313,$$

then $Z(\text{EXT})(w) = 20$, since

$$
\begin{align*}
\text{EXT}(w[1, 2]) &= \text{INV}(211221) = 5 \\
\text{EXT}(w[1, 3]) &= \text{INV}(113313) = 2 \\
\text{EXT}(w[1, 4]) &= \text{INV}(11441) = 2 \\
\text{EXT}(w[2, 3]) &= \text{MAJ}(232233) = 2 \\
\text{EXT}(w[2, 4]) &= \text{INV}(24422) = 4 \\
\text{EXT}(w[3, 4]) &= \text{INV}(43433) = 5
\end{align*}
$$

We shall find that EXS is not $Z$-Mahonian while EXT is $Z$-Mahonian.

Suppose $s$ is a binary family of splittable Mahonian statistics of type $\rho$. Therefore, for each word $b$ of type $x y^i$, $i \leq \rho_x$ and $j \leq \rho_y$, there is a "ranking" of $x$ and
y. That is, for each word of the form \( axb \), the contribution of \( x \), \( \alpha_x \), will depend upon \( b \), and will be either the number of \( y \)'s in \( a \) (if \( y \) beats \( x \)) or 0 (if \( x \) beats \( y \)). Similarly, for each word of the form \( ayb \), the contribution of \( y \), \( \alpha_y \), will be either the number of \( x \)'s in \( a \) (if \( x \) beats \( y \)) or 0 (if \( y \) beats \( x \)).

Now suppose \( v \) is a word of type \( \mu \leq \rho \). Each \( v[x, y] \) determines a relative ranking of \( x \) and \( y \), given by \( s \). Together, these rankings form a tournament, which we call \( T_v \).

**Theorem 2.** Suppose \( s \) is a binary family of splittable Mahonian statistics of type \( \rho \). Then \( s \) is Z-Mahonian on type \( \rho \) if and only if for each \( v \) of type \( \mu \leq \rho \), \( T_v \) is transitive on the letters in \( \rho - \mu \).

**Proof.** If \( T_v \) is transitive on the letters which appear in \( \rho - \mu = v \), then it corresponds to a permutation \( \pi_v \) on these letters, and is exactly the permutation required by Theorem 1 for \( Z(s) \) to be a splittable Mahonian statistic.

Conversely, suppose \( Z(s) \) is splittable Mahonian. By Theorem 1, the tournament \( T_v \) must be transitive. \( \square \)

Note that it is possible for \( s \) to be Z-Mahonian on some types, but not all types. Notable cases of splittable Mahonian binary families which are not Z-Mahonian for some types include D\( E \)N, RAW and KAD.

It is possible to describe how splittable statistics \( Z(s) \) can be built up. Suppose \( T \) is a transitive tournament. A transitive tournament \( S \) such that only team \( x \) changes positions relative to the remaining teams is called a reallocation of \( x \) in \( T \).

**Theorem 3.** Suppose \( s \) is a binary family of splittable Mahonian statistics of all types. Then each \( T_{sx} \) is a reallocation of \( x \) in \( T_v \) if and only if \( s \) is Z-Mahonian on all types.

**Proof.** The “only if” part is immediate. For the “if” part, since \( Z(s) \) is splittable on all types, by Theorem 2, \( T_v \) is transitive for each \( v \). But all the 2-letter subwords of \( xv \) which do not involve \( x \) will be the same as the 2-letter subwords of \( v \), so the order of these letters in \( T_{sx} \) will be the same as the order of these letters in \( T_v \). Therefore, \( T_{sx} \) will be a reallocation of \( x \) in \( T_v \). \( \square \)

5. Examples

In this section we will illustrate Theorem 2 and Theorem 3 with three basic examples. We can form other examples by combining these three basic examples in various ways.

Our first example is based on \( \text{MAJ} \) and the Zeilberger-Bressoud statistic \( \text{ZLB} \). Using Han’s encoding of \( \text{MAJ} \) [11], if the binary family is \( \text{MAJ} \), then \( T_w \) is defined as follows:

\( x \) beats \( y \) if and only if the leftmost \( x \) in \( w \) appears to the left of the leftmost \( y \).

To deal with cases where not all letters appear in \( w \), we concatenate a trailing \( K \ K - 1 \ldots 1 \) to \( w \). It is immediate from this description that \( T_w \) is transitive; in fact, \( \pi \) is the permutation of leftmost occurrences.

There are now several possible modifications. For example, instead of the leftmost \( x \) and \( y \), we could choose the \( x \) which is \( p_x \) positions from the left, where \( p_x \)
is a nonnegative integer for each \( x \) (and similarly for \( y \)). Or we could choose the rightmost \( x \) and \( y \).

The second example arises from a statistic related to work of Simion and Stanton [20]. Suppose \( w \) is a word using letters 1 and 2. An inversion in \( w \), \( w_i = 2, w_j = 1 \), with \( i < j \), is of type A if there is a 1 to the left of \( w_i \). It is of type B if, for every \( i < k < j \), \( w_k = 2 \). Then \( \text{sss}(w) \) is the number of type A inversions plus the number of type B inversions. Note that some inversions get counted twice, while others not at all. For example, if

\[
\text{w} = 2212111221211,
\]

then \( \text{sss}(w) = 20 \) since there are 14 type A inversions and 6 type B inversions.

This can be translated into an obvious splittable encoding as follows. For a word \( w \) using letters 1’s and 2’s, 1 beats 2 if and only if 1 appears in \( w \). In the above example, the rightmost 1 contributes 6 to the statistic (2 beats 1), but thereafter 1 beats 2, so the reverse inversions are counted (14 of them).

For a word \( w \) of arbitrary type \( \rho \), the tournament \( T_w \) will have, for \( x < y \), \( x \) beats \( y \) if and only if \( x \) appears in \( w \). This tournament is clearly transitive. The permutation \( \tau \) will first list all the letters appearing in \( w \) in increasing order, then all the letters not appearing in \( w \) in decreasing order.

Once again, many variations are possible. For example, instead of the first appearance of \( x \), the \( p_x \) appearance could be used. Or, as letters appear in \( w \), they could become losers instead of winners.

The final example is the interpolating statistic hinted at in [22]. Call a subset \( A \subseteq [K] \) contiguous if it has no “holes”; that is, if \( x, y \in A \) and \( x < z < y \), then \( z \in A \). Suppose \( A_1, \ldots, A_m \) is a collection of disjoint contiguous subsets. Now define \( \text{wht}(w) \) on words using \( x \)'s and \( y \)'s as follows. If \( x \) and \( y \) are in the same \( A_i \), then \( \text{wht}(w) = \text{maj}(w) \). Otherwise, \( \text{wht}(w) = \text{inv}(w) \).

Notice that if \( m = 0 \), then \( \text{wht} = \text{inv} \) and \( Z(\text{wht}) = \text{inv} \), while if \( m = 1 \) and \( A_1 = [K] \), then \( \text{wht} = \text{maj} \) and \( Z(\text{wht}) = \text{maj} \).

**Theorem 4.** The statistic \( \text{wht} \) is \( Z \)-Mahonian.

**Proof.** The tournament \( T_w \) is as follows. If \( x \) and \( y \) are both in the same \( A_i \), then \( x \) beats \( y \) if and only if \( x \) appears to the left of \( y \) in \( w \). Otherwise, \( x \) beats \( y \) if and only if \( x > y \). This tournament is clearly transitive. \( \square \)

Once again, many variations are possible. Instead of \( \text{maj} \), one of the statistics related to \( \text{maj} \) described above may be used. Or the statistic \( \text{sss} \) may be used. Or \( \text{maj} \) on some of the \( A_i \) and \( \text{sss} \) on other \( A_i \).

We now return to the two examples given in Section 4, \( \text{exs} \) and \( \text{ext} \). In the case of \( \text{exs} \), if \( \text{w} = 23 \), then in \( T_w \) 2 beats 3, 3 beats 4, and 4 beats 2.

In the case of \( \text{ext} \), the following table summarizes the situation. The first three columns are the possible mod 2 classes for \( p_2 \), \( p_3 \) and \( p_4 \). The next three columns give the statistic produced by \( \text{ext} \). The last column describes \( Z(\text{ext}) \).
<table>
<thead>
<tr>
<th>( \rho_2 )</th>
<th>( \rho_3 )</th>
<th>( \rho_4 )</th>
<th>( 23 )</th>
<th>( 24 )</th>
<th>( 34 )</th>
<th>( Z(\text{EXT}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>MAJ</td>
<td>MAJ</td>
<td>MAJ</td>
<td>( \text{WHT, } A_1 = {2, 3, 4} )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>MAJ</td>
<td>INV</td>
<td>INV</td>
<td>( \text{WHT, } A_1 = {2, 3} )</td>
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<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>INV</td>
<td>INV</td>
<td>INV</td>
<td>( \text{WHT, } A_1 = {3, 4} )</td>
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<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>INV</td>
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<td>( \text{WHT, } A_1 = {3, 4} )</td>
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<tr>
<td>0</td>
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<td>1</td>
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<td>( \text{WHT, } A_1 = {2, 3} )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>MAJ</td>
<td>MAJ</td>
<td>MAJ</td>
<td>( \text{WHT, } A_1 = {2, 3, 4} )</td>
</tr>
</tbody>
</table>

A few calculations will illustrate the frequency of splittable Mahonian statistics which are \( Z \)-Mahonian.

For type \((2, 1, 1)\), there are 32 binary families of splittable Mahonian statistics. Of these, 18 are \( Z \)-Mahonian.

For type \((2, 2, 1)\), there are 512 binary families of splittable Mahonian statistics. Of these, 138 are \( Z \)-Mahonian.

For type \((2, 2, 2)\), there are 32768 binary families, with 1122 \( Z \)-Mahonian.

6. Counting splittable statistics

In this section, we derive a generating function for the log of the number of splittable Mahonian statistics.

Given the simple recursive definition of a splittable Mahonian statistic, it is not surprising that we can find a recursion for the number of splittable statistics. Let \( f_\rho \) be the number of splittable Mahonian statistics on words of type \( \rho \). We have \( a(\rho)! \) different choices for the permutation \( \pi \), and for each of these and for each possible letter \( x \) ending the word we have \( f_{\rho \rightarrow x} \) possible splittable Mahonian statistics on the shorter word. Putting this together, we get

**Theorem 5.**

\[
f_\rho = (a(\rho)!) \prod_{x \in \rho} f_{\rho \rightarrow x}.
\]

Letting \( g_\rho = \log f_\rho \), we have

\[
g_\rho = \log(a(\rho)!) + \sum_{x \in \rho} g_{\rho \rightarrow x}.
\]

We can then find the generating function for the \( g_\rho \). Let \( x = \{x_1, x_2, \ldots\} \) be an infinite set of indeterminates and let \( t \) be another indeterminate. For a given type \( \rho \), let

\[
x^\rho = \prod_{i \geq 1} x_i^{e_i^\rho}.
\]

Now let

\[
G(x; t) = \sum_\rho g_\rho x^\rho t^{|\rho|}.
\]

**Theorem 6.**

\[
G(x; t) = \sum_{n \geq 0} H_n(x; t) \sum_{m=0}^{n} \binom{n}{m} (-1)^m \log((n - m)!) ,
\]
where
\[ H_n(x; t) = \frac{t^n e_n(x)}{1 - t c_1(x)} \prod_{j \geq 1} \frac{1}{1 - x_j t}, \]
and \( e_n(x) \) is the \( n \)-th elementary symmetric function in the variables \( x \).

Two interesting special cases can be computed exactly. First, when the words have two letters, we have this corollary.

**Corollary 7.** The coefficient of \( x_1^i x_2^j t^{i+j} \) in \( G(x; t) \) is
\[ \binom{i+j}{i} - 1 \log 2. \]

Corollary 7 has a simple combinatorial proof. If the list of words using \( i \) 1’s and \( j \) 2’s is written in a natural recursive way, that is, with the reverse ordered lexicographically, then between any pair of adjacent words, \( v \) and \( w \), in this list, there is a unique location \( k \) (from right to left) where they first differ, with \( v \) having a 1 and \( w \) having a 2. There are 2 possible tournament choices for this \( k \). Since there are \( \binom{i+j}{i} \) words in the list, there are \( \binom{i+j}{i} - 1 \) such adjacent pairs and therefore
\[ 2^{\binom{i+j}{i} - 1} \]
possible statistics.

For permutations, we have this result.

**Corollary 8.** The coefficient of \( x_1 \ldots x_n t^n \) in \( G(x; t) \) is
\[ \sum_{k=2}^{n} \binom{n}{k} (n-k)! \log(k!). \]

**proof of Theorem 6.** We first prove a finite version. Let
\[ G_k(x_1, \ldots , x_k; t) = G(x; t)|_{x_j=0, j>k}, \]
and
\[ H_k,n(x_1, \ldots , x_k; t) = H_n(x; t)|_{x_j=0, j>k}. \]

For a subset \( A \) of letters, let \( x[A] \) denote the list \( \{x_i \mid i \in A \} \) and \( x_A = \prod_{i \in A} x_i \).

We will show
\[ \hat{G}_k(x_1, \ldots , x_k; t) = \sum_{n=0}^{k} H_k,n(x_1, \ldots , x_k; t) \sum_{m=0}^{n} \binom{n}{m} (-1)^m \log((n-m)!)) \]
satisfies the recursion (2). Since the initial conditions are trivially satisfied, we must have \( \hat{G}_k = G_k \). Letting \( k \to \infty \) gives the theorem.

Recursion (2) and the definition of \( G \) translate into this identity:
\[ \sum_{j=0}^{k} (-1)^j \sum_{\substack{A \subseteq [k] \\mid \# A = k - j \}} (1 - t c_1(x[A])) H_{k-j,n}(x[A]; t) = t^k e_k(x) \prod_{i=1}^{k} \frac{1}{1 - x_i t}. \]

Replace \( G \) with \( \hat{G} \) into the left hand side of (4) and substitute (3), giving
\[ \sum_{j=0}^{k} (-1)^j \sum_{\substack{A \subseteq [k] \\mid \# A = k - j \}} (1 - t c_1(x[A])) H_{k-j,n}(x[A]; t) \sum_{m=0}^{n} \binom{n}{m} (-1)^m \log((n-m)!)) \]
If we now substitute for $H$ and simplify, then the coefficient of
\[ (-1)^{k+m} \log(m!) \prod_{i=1}^{k} \frac{1}{1-x_it} \]
is
\[ \sum (-1)^{|A|+|B|+|C|+|D|} x_{B,D}, \]
where the sum is over subsets $A$, $B$, $C$, and $D$ satisfying $C \subseteq B \subseteq A \subseteq [k]$, $D \subseteq A^c$, and $|C| = m$. It is easily seen that the coefficient of
\[ t^{|V|} x_U \]
in this expression is 0 unless $C = B = A = [k]$, $D = \emptyset$, and $m = k$, in which case the right-hand side of Equation 3 emerges.

We have not been able to derive a similar formula for all $Z$-Mahonian statistics of a given type. However, in the special case of permutations, we have this corollary of Theorem 2.

**Corollary 9.** The number of $Z$-Mahonian statistics on permutations of length $n$ is $n!$.

**Proof.** We count the number of binary families which yield $Z$-Mahonian statistics under Theorem 2. For the empty word, we have $n!$ possible transitive tournaments. Let $T$ be one such tournament. Now suppose $w$ is a permutation ending in $v$ and suppose that $v$ is not the empty word. Then $T_v$ is $T$ with the letters appearing in $v$ reassigned. But the letters remaining in $w$ do not appear in $v$, so these reassignments do not affect the original arrangement of the remaining letters. That arrangement is the same as in the initial $T$. \qed

**7. Remarks**

In [10] a new statistic, $lp$, was introduced. This statistic is notable for two reasons. First, it is splittable (though not Mahonian). Second, its distribution generating function is the same as the generating function for *charge* on words. Charge is the statistic discovered by Lascoux and Schützenberger [14] to resolve the Foulkes conjecture on the Kostka-Foulkes polynomials. For further details about charge, see [15].

Many of the results of this paper carry over to arbitrary splittable statistics with permutationally distributable generating functions (see [10] for definitions), as long as there is a generating statistic family (such as INV in the case of Mahonian statistics). In particular, it was conjectured in [10] that $lp$ is such a generating statistic family, and this has since been proved by the second author. Therefore, not only can $lp$ can be used to construct charge-distributed analogs of $maj$, $den$ and other splittable mahonian statistics, but a theorem similar to Theorem 2 can be stated to describe when the $Z$-operator produces a charge-distributed statistic. In particular, there are charge distributed analogs of both $maj$ and $zlb$.

As a direction for further investigation, we note that several authors (see, for example, [1], [4], [5], [6], [18], [19], and [21]) have constructed statistics on signed permutations analogous to Mahonian statistics. These constructions invites the question: Can any of them be extended to “signed words?” If so, is there a concept corresponding to a splittable statistic and an analogous $Z$ operator?
Finally, we suspect that, except for perhaps some sporadic cases, the only binary families which are splittable Mahonian are Z-Mahonian. However, we have no evidence and no theorems to this effect.

REFERENCES

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