1 Stieltjes transform

From last time, we see that to prove the semicircle law, it suffices to show that for all $z$ in the upper half-plane, $s_n(z) \to s_{\mu_{sc}}(z)$ almost surely. By directly controlling $s_n(z)$, the Stieltjes transform method can be used to find the semicircle law, even if we do not know this law in advance.

Let $z = a + bi$, $b > 0$. The main idea is to compare $s_n(z)$ to $s_{n-1}(z)$. Recall the Cauchy interlacing law, which says for any $n \times n$ Hermitian matrix with $(n-1) \times (n-1)$ minor $A_{n-1}$, one has $\lambda_i(A_n) \leq \lambda_i(A_{n-1}) \leq \lambda_{i+1}(A_n)$ for all $i = 1, \ldots, n-1$. The following “alternating” sum

$$\sum_{j=1}^{n-1} b \left( \frac{\lambda_j(M_n-1)}{\sqrt{n-1} - a} \right)^2 + b^2 - \sum_{j=1}^{n} b \left( \frac{\lambda_j(M_n)}{\sqrt{n} - a} \right)^2 + b^2$$

is bounded in $n$, because the function $x \mapsto b \left( \frac{x}{x-a} \right)^2$ has finite total variation, and

$$\{\lambda_1(M_n)/\sqrt{n}, \lambda_1(M_{n-1})/\sqrt{n}, \ldots, \lambda_{n-1}(M_{n-1})/\sqrt{n}, \lambda_n(M_n)/\sqrt{n}\}$$

forms a partition. Note that the above sum is the imaginary part of

$$\sqrt{n(n-1)}s_{n-1} \left( \frac{\sqrt{n}}{\sqrt{n-1}}(a + bi) \right) - ns_n(a + bi).$$

To see this, note that for instance

$$\sqrt{n(n-1)}s_{n-1} \left( \frac{\sqrt{n}}{\sqrt{n-1}}(a + bi) \right) = \sqrt{n(n-1)} \frac{1}{n-1} \sum_{j=1}^{n-1} \frac{1}{\lambda_j(M_{n-1}) - \frac{\sqrt{n}}{\sqrt{n-1}}(a + bi)}$$

$$= \sum_{j=1}^{n} \frac{1}{\lambda_j(M_{n-1})/\sqrt{n} - (a + bi)}.$$ 

The real part can be bounded similarly. Therefore, we see that

$$\sqrt{n(n-1)}s_{n-1} \left( \frac{\sqrt{n}}{\sqrt{n-1}}(a + bi) \right) - ns_n(a + bi) = O(1)$$

as $n \to \infty$. Also, note that the Stieltjes transform is smooth and $\frac{\sqrt{n}}{\sqrt{n-1}} - 1 = O(1/n)$, and so $s_{n-1} \left( \frac{\sqrt{n}}{\sqrt{n-1}}(a + bi) \right)$ and $s_{n-1}(a + bi)$ are close, and hence we can conclude that

$$s_n(a + bi) = s_{n-1}(a + bi) + O \left( \frac{1}{n} \right). \tag{1.1}$$

In particular, we see that $s_n$ is stable in $n$. Moreover, the right hand side depends only on the top left $(n-1) \times (n-1)$ minor, and it is independent of the $n$-th row and $n$-th column of the matrix.

We would like to know what $s_n$ converges to. Recall McDiarmid’s inequality:
Theorem 1.1. Let $X_1, \ldots, X_n$ be independent random variables taking values in ranges $R_1, \ldots, R_n$, and let $F : R_1 \times \cdots \times R_n \to \mathbb{C}$ be a function having bounded differences. That is, there exist constants $c_1, \ldots, c_n$ such that for all $i$,

$$|F(x_1, \ldots, x_n) - F(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)| \leq c_i$$

for all $x_i, x_i' \in R_i$. Then for any $\lambda > 0$, one has

$$\mathbb{P}(|F(X) - \mathbb{E}F(X)| \geq \lambda \sigma) \leq C \exp(-c\lambda^2)$$

for some $C, c > 0$, where $\sigma^2 := \sum_{i=1}^n c_i^2$.

Note that the $X_j$'s above can be vectors of different lengths. Now, observe that (1.1) still holds if we resample the $n$-th row and the $n$-th column. Write $s'_n(z)$ for the Stieltjes transform of the resampled matrix. Then by (1.1), we have $s_n(z) = s'_n(z) + O(1/n)$. Moreover, if we interchange the $n$-th row and $j$-th row, and also $n$-th column and $j$-th column, so that the matrix is still Hermitian and has the same distribution as before, the Stieltjes transform will remain the same. In other words, if we resample the $j$-th row (and hence the $j$-th column by symmetry), the change of $s_n(z)$ is at most $O(1/n)$. Therefore, applying McDiarmid's inequality with $X_j = (\xi_{j,j}, \xi_{j,j+1}, \ldots, \xi_{j,n})$, we have

$$\mathbb{P}(|s_n(z) - \mathbb{E}s_n(z)| \geq c' \lambda/\sqrt{n}) \leq C \exp(-c\lambda^2).$$

Take $\lambda = n^{1/3}$ and apply Borel-Cantelli, we obtain that almost surely,

$$s_n(z) - \mathbb{E}s_n(z) \leq O(n^{-1/6})$$

for all large $n$. Therefore, almost surely, $s_n(z) - \mathbb{E}s_n(z) \to 0$ for all $z$ in the upper half-plane.

Thus, it remains to study what $\mathbb{E}s_n(z)$ tends to as $n \to \infty$. Note that

$$\mathbb{E}s_n(z) = \frac{1}{n} \sum_{j=1}^n \mathbb{E} \left[ \left( \frac{1}{\sqrt{n}} M_n - zI_n \right)_{j,j}^{-1} \right]$$

Again, interchanging the rows and columns so that the matrix $M_n$ is still Wigner will not alter the distribution of the matrix. In particular, the $(j, j)$-th entry of $\left( \frac{1}{\sqrt{n}} M_n - zI_n \right)^{-1}$ has the same distribution as the $(n, n)$-th entry. Therefore,

$$\mathbb{E}s_n(z) = \frac{1}{n} \sum_{j=1}^n \mathbb{E} \left[ \left( \frac{1}{\sqrt{n}} M_n - zI_n \right)_{j,j}^{-1} \right] = \mathbb{E} \left[ \left( \frac{1}{\sqrt{n}} M_n - zI_n \right)_{n,n}^{-1} \right].$$

So we need only to study the last entry of $\left( \frac{1}{\sqrt{n}} M_n - zI_n \right)^{-1}$, and we will use a formula based on the Schur complement.
References
