GLOBAL EXISTENCE AND UNIQUENESS OF A THREE-DIMENSIONAL MODEL OF CELLULAR ELECTROPHYSIOLOGY

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Abstract. We study a three-dimensional model of cellular electrical activity, which is written as a pseudodifferential equation on a closed surface Γ in $\mathbb{R}^3$ coupled with a system of ordinary differential equations on Γ. Previously the existence of a global classical solution was not known, due mainly to the lack of a uniform L^∞ bound. The main difficulty lies in the fact that, unlike the Laplace operator that appears in traditional models, the pseudodifferential operator in the present model does not satisfy the maximum principle. We overcome this difficulty by introducing the notion of "quasipositivity principle" and prove a uniform L^∞ bound of solutions – hence the existence of global classical solutions – for a large class of nonlinearities including the FitzHugh-Nagumo and the Hodgkin-Huxley kinetics. We then study the asymptotic behavior of solutions to show that the system possesses a finite dimensional global attractor consisting entirely of smooth functions despite the fact that the system is only partially dissipative. We also show that ordinary differential equation models without spatial extent, often used in modeling studies, can be obtained from the present model in the small-cell-size limit.

1. Introduction

In this paper we study the well-posedness of a system of partial differential equations (PDE) that governs the electrical activity of biological cells. Consider a smooth bounded domain Ω_i in $\mathbb{R}^3$ with a finite number of connected components. Denote by Γ the boundary of Ω_i. The region Ω_i represents the interior of a cell or a finite collection of cells. The boundary Γ corresponds to the cellular membrane. We let $\Omega_e = \mathbb{R}^3\backslash(\Omega_i \cup \Gamma)$ denote the extracellular space, and suppose that Ω_e is connected.

$$\nabla \cdot \sigma_i \nabla v_i = 0 \text{ in } \Omega_i, \tag{1.1a}$$

$$\nabla \cdot \sigma_e \nabla v_e = 0 \text{ in } \Omega_e, \tag{1.1b}$$

$$\sigma_i \frac{\partial v_i}{\partial \mathbf{n}} = \sigma_e \frac{\partial v_e}{\partial \mathbf{n}} \text{ on } \Gamma, \tag{1.1c}$$

$$C_m \frac{\partial v}{\partial t} - f(v, w_1, \cdots, w_N) = -\sigma_i \frac{\partial v_i}{\partial \mathbf{n}},\ v \equiv v_i - v_e \text{ on } \Gamma, \tag{1.1d}$$

$$\frac{\partial w_k}{\partial t} = g_k(v, w_1, \cdots, w_N),\ k = 1 \cdots N, \text{ on } \Gamma, \tag{1.1e}$$

$$v_e \to 0 \text{ as } |x| \to \infty. \tag{1.1f}$$

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Here, v_i and v_e denote the electrostatic potential, defined in Ω_i and Ω_e, respectively; σ_i and σ_e are the electrical conductivities of the intracellular and extracellular spaces. In general, σ_i and σ_e are 3×3 symmetric positive definite matrices that depend on position, but in applications, they are often taken to be positive scalars that are spatially constant within each region Ω_i and Ω_e [14]. This is the case we treat in this paper.

Equations (1.1a) and (1.1b) are statements of ohmic current continuity, while equation (1.1c) expresses current continuity across the cell membrane interface. The cell membrane acts primarily as a capacitor, but it also has ion channels, through which electric current may pass. Electric current $-\sigma_i \frac{\partial v_i}{\partial \mathbf{n}}$ that hits the membrane from the Ω_i side of the membrane Γ either goes across the membrane through ion channels or contributes to the capacitive surface charge on the membrane. The capacitive current is given by $C_m \frac{\partial v}{\partial t}$ where C_m is the capacitance per unit area and $v = v_i - v_e$ is the *membrane potential* which represents the jump in the electrostatic potential across the membrane. The ion channel current is given by the function $f(v, w_1, \cdots, w_n)$. This function expresses the current voltage relationship for the (group of) ion channels found on the membrane. The variables $w_1, \cdots, w_N$ are called *gating variables*, and are defined solely on the membrane. They describe the biophysical state of the ionic channels, and evolve according to a system of ordinary differential equations (ODE) (1.1e). Representative examples for the functions f and g_k include the cubic FitzHugh-Nagumo and Hodgkin-Huxley kinetics [14].

The above model is a natural extension of the cable model, a standard model of cellular electrical activity, in which equations (1.1a), (1.1b), (1.1c), (1.1d) are replaced by a single one-dimensional semilinear heat equation [14]. Since the system (1.1a)-(1.1f) is posed in three dimensions, this model makes it possible to study the effect of three-dimensional cellular geometry on electrophysiology. For this reason, the system (1.1a)-(1.1f) is called the *3D cable model* [6].

The 3D cable model has been used to study various three-dimensional effects in electrophysiology [29, 6, 4]. More recent applications include the study of ephaptic effects in the central nervous system [13, 10]. Related models have also been used to study the development of algae [17], and anomalous action potential propagation in cardiac tissue [24, 23].

The 3D cable model may be considered a fundamental equation of cellular electrophysiology in the sense that many of the important models of electrophysiology may be derived from it. The cable model mentioned above and the bidomain model which is the standard model for electrocardiology can both be derived as appropriate limits of the 3D cable model [25, 28].

From a mathematical point of view, the 3D cable model consists of a semilinear parabolic evolution equation for the electrostatic potential v coupled with ODEs for the gating variables w_k. This is similar to the cable model, except that the principal linear differential operator is not the second order derivative $-\frac{\partial^2 \cdot}{\partial x^2}$, but comes in the form of boundary conditions for an elliptic problem (see equation (1.1c)). This operator can be expressed as a pseudodifferential operator Λ defined on the membrane Γ whose properties are similar to the Dirichlet-Neumann map for bounded domains.

Only a limited number of mathematically rigorous studies have been made on the 3D cable model. Those studies include [9, 2, 36]. In [2], the authors consider the single equation case in which the parabolic evolution equation is not coupled to ODEs. The authors use an L^2 framework to prove existence and uniqueness of global weak solutions when the nonlinear terms are globally Lipschitz. In [36], the author considers the system in which the parabolic evolution equation is coupled with ODEs. Extending the methods in [2], the author proves local existence and uniqueness for short times and global existence of weak solutions with some restrictions on the nonlinear terms using a Schauder fixed point argument.

From a biophysical standpoint, it is natural to expect much more to be true for this system. Indeed, since the 3D cable model may be seen as a three dimensional generalization of the cable model, we expect many of the known properties of the cable model to hold for the 3D cable model. For example, assume FitzHugh-Nagumo or Hodgkin-Huxley kinetics for f and g_k in (1.1d) and (1.1e). In addition to global in time existence and uniqueness, we expect solutions to be uniformly bounded in time in the L^∞ norm, and the solutions to be classical under suitable regularity conditions on the initial data [30]. The difficulty with the 3D cable model is that the operator Λ does not, in general, satisfy the maximum principle. This precludes the direct application of the method of invariant regions – or invariant rectangles – used successfully for the cable model [30].

In this paper, we show that the above biophysical expectations are indeed correct. We prove global existence and uniqueness of solutions under mild conditions on the nonlinear terms. Moreover, we prove a global uniform bound on the solutions in the L^∞ norm. The solutions are classical under suitable assumptions on the initial data.

The outline of the paper is as follows. In Section 2, we rewrite our system in terms of parabolic evolution equations on the membrane Γ. We study the linear semigroup generated by $-\Lambda$ in Sections 3 and 4. We first establish estimates for $\exp(-t\Lambda)$ in L^2 based Sobolev spaces. To study the properties of $\exp(-t\Lambda)$ in L^∞, we introduce the notion of *quasipositivity*. Roughly speaking, Λ satisfies the quasipositivity principle if it can be written as a bounded perturbation of a generator of a positive semigroup. We shall then show that Λ is indeed a quasipositive operator using the method of layer potentials and an "interpolation" argument. Semigroup generation in L^p based Sobolev spaces is obtained by interpolating between the L^2 and L^∞ results. In Section 5, we use semigroup theory to establish local existence of solutions. In Sections 6 and 7, we discuss global existence. Depending on the nonlinearities f and g, we discuss two cases, systems of FitzHugh-Nagumo type and of Hodgkin-Huxley type. In Section 6, we adapt method of invariant rectangles to establish global existence and uniform L^∞ bounds for FitzHugh-Nagumo type systems. Here, the quasipositivity of the operator plays a crucial role. In Section 7, we establish global existence for Hodgkin-Huxley type systems using energy methods.

In Section 8, we study asymptotic smoothing and global attractors. Given that w in (1.1e) only satisfies an ODE, we cannot expect immediate smoothing of w for positive time. However, if the nonlinearity g in (1.1e) satisfies $\frac{\partial g}{\partial w} < 0$, then w will acquire the smoothness of v after an infinite time, from which we can conclude that any point on the ω-limit set is in fact smooth. We show that the above system possesses a global attractor that consists entirely of smooth functions. We shall also show that the global attractor has finite Hausdorff dimension. The methods used here can be easily adapted to study partially dissipative second order parabolic problems on a finite domain (for example, the cable model on a finite domain), to generalize the results of [22].

As mentioned above, many important models of cellular electrophysiology may be derived as formal limits of the 3D cable model. The simplest such model is the point model in which v and w_k are assumed spatially constant on the membrane Γ and satisfy a system of ODEs [14, 15]. In the final section, we show that v and w_k converge exponentially in time to their respective spatial averages in the L^∞ norm if the cell is sufficiently small (or equivalently, if σ_i and σ_e are sufficiently large). Here it is worth noting that the required smallness of the cell for the above convergence to take place depends largely on the geometrical shape of the cell.

This paper is an initial step toward a more complete analytical understanding of the 3D cable model and its relationship with other important models in electrophysiology. In a forthcoming paper, we plan to address well-posedness of the 3D cable model on unbounded cylindrical domains and its relation with the cable model. We also hope to extend our results here to the case when there are multiple cells, i.e., when Ω_i consists of finitely many connected components. We hope that the notion of quasipositivity introduced in this paper will find use in the analysis of other PDE systems that do not satisfy the maximum principle.

2. Problem Setup

As stated at the beginning of the Introduction, we shall be concerned with system (1.1):

$$\Delta v_i = 0 \quad \text{in } \Omega_i, \tag{2.1a}$$
$$\Delta v_e = 0 \quad \text{in } \Omega_e, \tag{2.1b}$$
$$\frac{\partial v_i}{\partial \mathbf{n}} = \sigma \frac{\partial v_e}{\partial \mathbf{n}} \quad \text{on } \Gamma, \tag{2.1c}$$
$$\frac{\partial v}{\partial t} - f(v,w) = -\frac{\partial v_i}{\partial \mathbf{n}}, \quad v \equiv v_i - v_e \quad \text{on } \Gamma, \tag{2.1d}$$
$$\frac{\partial w}{\partial t} = g(v,w) \quad \text{on } \Gamma, \tag{2.1e}$$
$$v_e(x) \to 0 \quad \text{as } |x| \to \infty, \tag{2.1f}$$

where $w = (w_1, \cdots, w_N)$ and $g = (g_1, \cdots, g_N)$ (see Fig. 1). We treat the case in which the electrical conductivities are spatially constant within each domain Ω_i and Ω_e. Equations (1.1a) and (1.1b) then reduce to the Laplace equations (2.1a) and (2.1b). We have written (1.1c) and (1.1d) as (2.1c), (2.1d), where $\sigma > 0$ is a constant, eliminating the constants C_m and σ_i by rescaling. The function f is a smooth function from $\mathbb{R}^{N+1}$ to $\mathbb{R}$ and g is a smooth function from $\mathbb{R}^{N+1}$ to $\mathbb{R}^N$. In Sections 6 and 9, we shall discuss qualitative conditions on the vector field $(f,g) = (f, g_1, \cdots, g_N)$ that ensures global existence of solutions. We specify initial values on Γ as follows

$$v = v^0,\ w = w^0 = (w_1^0, \cdots, w_N^0), \quad \text{for } t = 0 \quad \text{on } \Gamma. \tag{2.2}$$

We shall also consider the case $N = 0$ (the *single equation case*), in which v is the only unknown and (2.1e) is absent from the system of equations.

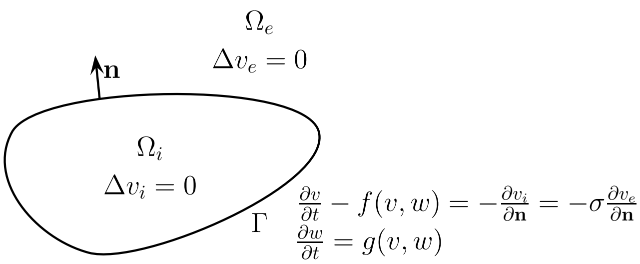

FIGURE 1. The membrane Γ separates the regions Ω_i and Ω_e, where v_i and v_e are harmonic (v denotes $v_i - v_e$ on Γ). The functions $w = (w_1, \cdots, w_N)$ are defined only on the membrane Γ.

In place of the above *two-phase* problem, we shall also consider the following *one-phase* problem, in which we only consider the equation for $v \equiv v_i$:

$$\Delta v_i = 0 \quad \text{in } \Omega_i, \tag{2.3a}$$

$$\frac{\partial v}{\partial t} - f(v,w) = -\frac{\partial v_i}{\partial \mathbf{n}}, \quad v = v_i \quad \text{on } \Gamma, \tag{2.3b}$$

$$\frac{\partial w}{\partial t} = g(v,w) \quad \text{on } \Gamma. \tag{2.3c}$$

It would sometimes be convenient to look at the one-phase problem as a formal limit of the two-phase problem as $\sigma \to \infty$. We shall consider the single equation case for the one-phase problem as well.

Both the one and two-phase problems can be rewritten as equations defined solely on the boundary Γ by introducing an operator Λ which we specify below. For the two-phase problem, given a smooth function u on Γ, we set

$$(E_\sigma u)(x) = \begin{cases} u_i(x) & \text{if } x \in \Omega_i, \\ u_e(x) & \text{if } x \in \Omega_e, \end{cases} \tag{2.4}$$

where the pair (u_i, u_e) is a solutions of the following boundary value problem:

$$\Delta u_i = 0 \quad \text{in } \Omega_i, \quad \Delta u_e = 0 \quad \text{in } \Omega_e, \tag{2.5a}$$

$$u_i - u_e = u, \quad \frac{\partial u_i}{\partial \mathbf{n}} = \sigma\frac{\partial u_e}{\partial \mathbf{n}} \quad \text{on } \Gamma, \tag{2.5b}$$

$$u_e(x) \to 0 \quad \text{as } |x| \to \infty. \tag{2.5c}$$

It is easily seen that such a pair (u_i, u_e) exists uniquely for any reasonably regular function u. The existence can be shown, for example, by setting $u_i = v^*|_{\Omega_i}$, $u_e = v^*|_{\Omega_e} - \tilde{u}$, where $\tilde{u}$ is the solution of the exterior problem

$$\Delta\tilde{u} = 0 \quad \text{in } \Omega_e, \qquad \tilde{u} = u \quad \text{on } \Gamma, \tag{2.6}$$

and v^* is the minimizer of the following functional:

$$I_\sigma(v) := \int_{\Omega_i} |\nabla v|^2 dx + \sigma\Big(\int_{\Omega_e} |\nabla(v - \tilde{u})|^2 dx\Big), \qquad v \in \dot{H}^1(\mathbb{R}^3). \tag{2.7}$$

Here $\dot{H}^1(\mathbb{R}^3)$ denotes, as usual, the Hilbert space obtained by taking the closure of $C_0^\infty(\mathbb{R}^3)$ with respect to the semi-norm $\int_{\mathbb{R}^3} |\nabla v|^2 dx$. The functional I_σ is strictly convex; hence it has no critical point other than the global minimizer v^*. This implies the uniqueness of the pair (u_i, u_e). (The uniqueness follows also from the maximum principle.) Thus $E_\sigma u$ is well-defined. We then define a family of operators depending on $0 < \sigma < \infty$:

$$\Lambda_\sigma u \equiv \frac{\partial u_i}{\partial \mathbf{n}} = \sigma \frac{\partial u_e}{\partial \mathbf{n}}. \tag{2.8}$$

For the one-phase problem we proceed as follows. Given a smooth function u on Γ, consider the boundary value problem:

$$\Delta u_i = 0 \ \text{ in } \Omega_i, \quad u_i = u \ \text{ on } \Gamma. \tag{2.9}$$

This problem clearly has a unique solution. Denote this solution by:

$$(E_\infty u)(x) = \begin{cases} u_i(x) & \text{if } x \in \Omega_i, \\ 0 & \text{if } x \in \Omega_e. \end{cases} \tag{2.10}$$

Here the solution is extended to Ω_e for later convenience. We then define:

$$\Lambda_i u = \frac{\partial u_i}{\partial \mathbf{n}}. \tag{2.11}$$

This operator is nothing but the Dirichlet-Neumann map for the region Ω_i.

As one can see from the variational formulation (2.7), the solution $E_\sigma u$ of system (2.5) converges to $E_\infty u$ as $\sigma \to \infty$, at least if u is sufficiently regular. Indeed, if we denote by v_σ^* the minimizer of I_σ in $\dot{H}^1(\mathbb{R}^3)$, and substitute into I_σ a function $\tilde{v}$ that coincides with $E_\infty u$ in Ω_i and $\tilde{u}$ in Ω_e, then

$$I_\sigma(v_\sigma^*) \le I_\sigma(\tilde{v}) = \int_{\Omega_i} |\nabla E_\infty u|^2 dx. \tag{2.12}$$

Since the right-hand side is independent of σ, we have

$$\int_{\Omega_e} |\nabla(v_\sigma^* - \tilde{u})|^2 dx = O(\sigma^{-1}) \to 0 \quad \text{as} \quad \sigma \to \infty. \tag{2.13}$$

It follows that $v_\sigma^*|_\Gamma \to u$ in $H^{1/2}(\Gamma)$, which implies $v_\sigma^* \to E_\infty u$ in Ω_i. Thus we have $E_\sigma u \to E_\infty u$ as $\sigma \to \infty$. This observation justifies the use of the suffix ∞ in $E_\infty u$. We shall sometimes use the notation Λ_∞ in place of Λ_i.

Setting $\Lambda = \Lambda_i, \Lambda_\sigma$, we can write system (2.3) or (2.1) as:

$$\frac{\partial v}{\partial t} = -\Lambda v + f(v, w), \tag{2.14a}$$

$$\frac{\partial w}{\partial t} = g(v, w). \tag{2.14b}$$

Here, the equations are defined on Γ. The $N = 0$ single equation case reduces to

$$\frac{\partial v}{\partial t} = -\Lambda v + f(v). \tag{2.15}$$

We note that it is relatively easy to extend the theory developed in this paper to a system in which the functions w_k also satisfy equations similar to v:

$$\frac{\partial w_k}{\partial t} = -\epsilon_k \Lambda w_k + g_k(v, w_1, \cdots, w_n), \tag{2.16}$$

where ϵ_k are positive constants. In fact, w_k will possess better regularity properties than when $\epsilon_k = 0$, since the term $-\epsilon_k \Lambda w_k$, as we shall see, has a regularizing effect similar to the diffusion operator. We shall not pursue the case $\epsilon_k > 0$ here, however, since this has no bearing in the context of electrophysiology.

Although we only defined Λ_i and Λ_σ for suitably smooth functions, their domain of definition can be enlarged to yield closed operators in certain function spaces, as we shall see below. When there is no room for confusion, we shall often abuse notation by referring to such extensions also as Λ_i and Λ_σ.

3. Operator Properties in Sobolev Spaces

In this section, we examine the properties of the operator Λ acting on Sobolev spaces $W_p^s(\Gamma), 1 < p < \infty, s \in \mathbb{R}$. We shall also discuss properties of the semigroup generated by $-\Lambda$ in the L^2 based interpolation spaces $H^s(\Gamma) = W_2^s(\Gamma), s \geq 0$. In Section 4.3, we shall discuss semigroup generation in $W_p^s(\Gamma)$.

We collect some elementary facts about layer potentials. We refer the reader to [8, 35]. We introduce some notation regarding pseudodifferential operators (ΨDO for short). The set of ΨDO on Γ of order $m \in \mathbb{R}$ will be denoted by $\Psi^m(\Gamma)$ (we shall only make use of the "classical" pseudodifferential operators whose symbol class is often denoted by $S_{1,0}^m$ [35, 8]). We also let $\Psi^{-\infty}(\Gamma) = \bigcap_{m\in\mathbb{R}} \Psi^m(\Gamma)$. We shall make use of the fact that a ΨDO of order m maps $B_p^s(\Gamma)$ to $B_p^{s-m}(\Gamma)$ for $s, m \in \mathbb{R}, 1 < p < \infty$.

Let ϕ be a sufficiently smooth function defined on the membrane Γ. Define the single layer potential with density ϕ by:

$$(S\phi)(x) = \int_\Gamma \frac{1}{4\pi |x-y|} \phi(y) dS_y, \tag{3.1}$$

where $x \in \mathbb{R}^3$ and dS_y denotes integration over Γ with respect to y. Given a smooth function u defined on $\mathbb{R}^3 \backslash \Gamma$, denote by $L_i u$ and $L_e u$ the limiting values of u from the Ω_i side and the Ω_e side of Γ, respectively, provided that these limits exist in a reasonable sense. Likewise, define $N_i u$ and $N_e u$ to be the normal derivatives of u on Γ evaluated from the Ω_i side and the Ω_e side of Γ, respectively. Here the unit normal is taken in the outward direction (pointing toward Ω_e) in both cases. We then have:

$$L_i(S\phi) = L_e(S\phi), \tag{3.2}$$

$$N_i(S\phi) = \frac{1}{2}\phi + T_S\phi, \tag{3.3}$$

$$N_e(S\phi) = -\frac{1}{2}\phi + T_S\phi, \tag{3.4}$$

$$(T_S\phi)(x) = -\int_\Gamma \frac{\mathbf{n}_x \cdot (x-y)}{4\pi |x-y|^3} \phi(y) dS_y, \; x \in \Gamma. \tag{3.5}$$

Here $\mathbf{n}_x$ is the unit outward normal at $x \in \Gamma$. Set

$$L_S\phi := L_i(S\phi) = L_e(S\phi).$$

Note that the leading-order singularity of the kernel of $L_S\phi$ given in (3.1) is of the form $|x-y|^{1-d}$, where $d = 2$ is the dimension of the surface Γ. This implies that L_S extends to an elliptic ΨDO of order -1 (see, for example, Chapter 7 Section 11 of [35]). The smoothness of Γ implies that the integration kernel K_S in (3.5) satisfies the following bound:

$$|K_S(x,y)| \leq \frac{C}{|x-y|} \tag{3.6}$$

where C is a constant that does not depend on x or y. This implies that T_S extends to a ΨDO of order -1.

Define the double layer potential with density μ by:

$$(D\mu)(x) = -\int_\Gamma \frac{\mathbf{n}_y \cdot (x-y)}{4\pi |x-y|^3}\mu(y)dS_y, \tag{3.7}$$

where $x \in \mathbb{R}^3 \backslash \Gamma$ and $\mathbf{n}_y$ is the unit normal vector at $y \in \Gamma$. We have:

$$L_i(D\mu) = \frac{1}{2}\mu + T_D\mu, \tag{3.8}$$

$$L_e(D\mu) = -\frac{1}{2}\mu + T_D\mu, \tag{3.9}$$

$$N_i(D\mu) = N_e(D\mu), \tag{3.10}$$

$$(T_D\mu)(x) = -\int_\Gamma \frac{\mathbf{n}_y \cdot (x-y)}{4\pi |x-y|^3}\mu(y)dS_y,\ x \in \Gamma. \tag{3.11}$$

We note that T_D is the negative transpose of T_S, and thus shares the same smoothing properties. The quantity $N_i(D\mu) = N_e(D\mu)$ can be identified with $\Lambda_1\mu$, since $D\mu$ satisfies the equations (2.5a)-(2.5c) with $\sigma = 1$. In the sequel, we shall make use of the well known fact that Λ_i generates a ΨDO of order 1 (see for example [35] or [16]).

We now examine the properties of Λ_σ acting on $W_p^s(\Gamma)$. Note first that the solution $E_\sigma u$ of (2.5a)-(2.5c) and $E_\infty u$ of (2.9) have the same jump in value across the boundary Γ. Thus, we seek a single layer density ϕ that satisfies

$$E_\sigma u = E_\infty u + S\phi. \tag{3.12}$$

By substituting this into (2.5b), we find the following expression:

$$N_i(S\phi) + \Lambda_\infty u = \sigma N_e(S\phi). \tag{3.13}$$

Substituting (3.4) and (3.3) we have:

$$\frac{1}{2}\phi + T_S\phi + \Lambda_\infty u = -\frac{\sigma}{2}\phi + \sigma T_S\phi. \tag{3.14}$$

Rearranging terms,

$$\phi - 2\epsilon T_S\phi = -(1-\epsilon)\Lambda_\infty u, \quad \epsilon = \frac{\sigma - 1}{\sigma + 1}. \tag{3.15}$$

Note here that $|\epsilon| < 1$. We now establish that $1 - 2\epsilon T_S$ has a bounded inverse in any $W_p^s(\Gamma), s \in \mathbb{R}, 1 < p < \infty$. Since $T_S \in \Psi^1(\Gamma)$, it is a bounded operator from $W_p^s(\Gamma)$ to $W_p^{s+1}(\Gamma)$. Given that Γ is a smooth closed compact surface, $W_p^{s+1}(\Gamma)$ is compactly embedded in $W_p^s(\Gamma)$ and hence, T_S is a compact operator on $W_p^s(\Gamma)$. Therefore, $1 - 2\epsilon T_S$ is an operator of Fredholm index 0, and has an inverse if and only if its null space is trivial. Suppose $\phi \in W_p^s(\Gamma)$ is in the null space of $1 - 2\epsilon T_S$. Then,

$$\phi = 2\epsilon T_S\phi.$$

By the mapping property of T_S, ϕ must be in $W_p^{s+1}(\Gamma)$. We conclude by induction that ϕ must be smooth. Next, we note that the function ϕ belongs to the null space if and only if (3.15) holds for some function u satisfying $\Lambda_\infty u = 0$, that is, if and only if (3.13) holds with $\Lambda_\infty u = 0$. From this, (3.2) and the Hopf boundary lemma, we see that $S\phi|_\Gamma$ cannot attain a positive maximum nor a negative minimum, which implies $S\phi = 0$; hence $\phi = 0$. The use of the Hopf boundary lemma is justified by the smoothness of ϕ. This establishes the invertibility of $1 - 2\epsilon T_S$ in $W_p^s(\Gamma)$. Thus we may invert (3.15):

$$\phi = -(1 - 2\epsilon T_S)^{-1}(1 - \epsilon)\Lambda_\infty u. \tag{3.16}$$

In fact, $(1 - 2\epsilon T_S)^{-1} \in \Psi^0(\Gamma)$, as can be seen as follows. Take a parametrix $Q_T \in \Psi^0(\Gamma)$ of $1 - 2\epsilon T_S$. Then we have the expression $Q_T(1 - 2\epsilon T_S) = 1 - R_T$ where R_T is a smoothing operator (meaning that it maps any distribution on Γ to a smooth function). For $u \in W_p^s(\Gamma)$, an easy calculation shows that

$$(1 - 2\epsilon T_S)^{-1}u = Q_T u + R_T(1 - 2\epsilon T_S)^{-1}u. \tag{3.17}$$

Since $R_T \in \Psi^{-\infty}(\Gamma)$, $R_T(1 - 2\epsilon T_S)^{-1}$ maps any $u \in W_p^s(\Gamma)$ to a smooth function. This implies that $R_T(1 - 2\epsilon T_S)^{-1}$ is a smoothing operator. Given $Q_T \in \Psi^0(\Gamma)$, $(1 - 2\epsilon T_S)^{-1} \in \Psi^0(\Gamma)$.

Combining (3.16) and (3.3), we may now express $\Lambda_\sigma \phi$ as:

$$\begin{aligned} \Lambda_\sigma u &= \Lambda_\infty u + N_i(S\phi) \\ &= \Lambda_\infty u - \frac{1}{2}(1 - \epsilon)(1 + 2T_S)(1 - 2\epsilon T_S)^{-1}\Lambda_\infty u. \end{aligned} \tag{3.18}$$

We may rewrite this as:

$$\Lambda_\sigma u = \frac{1 + \epsilon}{2}\Lambda_\infty u - (1 - \epsilon^2)(1 - 2\epsilon T_S)^{-1}T_S\Lambda_\infty u. \tag{3.19}$$

Notice that the second term is written as a composition of three ΨDO, Λ_∞, T_S and $(1 - 2\epsilon T_S)^{-1}$ which are respectively of order -1, 1 and 0. This means that the second term as a whole is a ΨDO of order 0 (note that composition of PsiDOs does not pose a problem since we are working on a compact surface). In particular, the second term is a bounded map from $W_p^s(\Gamma)$ to itself. Since $(1 + \epsilon)/2 > 0$ and is thus not equal to 0, we conclude that Λ_σ is an elliptic ΨDO of order 1. We end this section with the following result on semigroup generation in L^2 based Sobolev spaces.

Proposition 3.1. *Let $\Lambda = \Lambda_i, \Lambda_\sigma$. The operator Λ is a positive semidefinite self-adjoint operator on $L^2(\Gamma)$ with compact resolvent. The operator $-\Lambda$ therefore generates an analytic semigroup* $\exp(-t\Lambda)$ *on $H^s(\Gamma), s \in \mathbb{R}$.*

Proof. Given two smooth functions u, w defined on Γ and a constant $\mu \geq 0$, an easy computation using the definition Λ_σ and the Green's theorem yields the following:

$$\begin{aligned} \langle u, (\mu + \Lambda_\sigma)w\rangle &= \int_{\Omega_i} \nabla(E_\sigma u) \cdot \nabla(E_\sigma w)dx \\ &\quad + \int_{\Omega_e} \sigma\nabla(E_\sigma u) \cdot \nabla(E_\sigma w)dx + \int_\Gamma \mu uwdS_x, \end{aligned} \tag{3.20}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $L^2(\Gamma)$. This shows that $\mu + \Lambda_\sigma$ is symmetric and positive semidefinite for $\mu \geq 0$. Moreover, if $\mu > 0$, $\mu + \Lambda_\sigma$ is positive definite. Given that Λ_σ is an elliptic ΨDO of order 1, we see that Λ_σ is a self-adjoint positive semi-definite operator on $L^2(\Gamma)$ that maps $H^s(\Gamma) = W_2^s(\Gamma)$ to $H^{s-1}(\Gamma)$. Ellipticity also implies that $\mu + \Lambda_\sigma, \mu > 0$, maps $H^s(\Gamma)$ onto $H^{s-1}(\Gamma)$. Given that $H^s(\Gamma)$ is embedded compactly into $H^{s-1}(\Gamma)$, Λ has compact resolvent. A similar argument shows that Λ_i has the same properties. This is sufficient to show that $-\Lambda$ generates an analytic semigroup in the Sobolev spaces $H^s(\Gamma), s \in \mathbb{R}$ [27]. □

4. Quasipositivity and Semigroup Generation

We now examine quasipositivity and semigroup generation in $C(\Gamma)$, the space of continuous functions on Γ and the Sobolev spaces $W_p^s(\Gamma), s \geq 0, 1 < p < \infty$. Recall that we are assuming, throughout the present paper, that Ω_i is bounded, therefore Γ is a compact surface.

4.1. **The positivity and quasipositivity principles.** Consider a compact Hausdorff space K and let $C(K)$ denote the space of real-valued continuous functions on K endowed with the norm $\|u\|_{C(K)} = \max_{x\in K} |u(x)|$. For the purposes of this paper, all we need to discuss is the case $K = \Gamma$. However, in this subsection we treat the problem in a more general framework in order to clarify the underlying ideas and to make the results applicable to other problems.

We now define the positivity and quasipositivity principles for operators defined on $C(K)$.

Definition 4.1 (Positivity, quasipositivity and the nonpositivity index)**.** *Let L be a densely defined linear operator on $C(K)$ whose domain is given by $\mathcal{D}(L)$. L satisfies the* positivity principle *if, for any function $u \geq 0, u \in \mathcal{D}(L)$, the following holds at every point $x_0 \in K$ where $u(x_0) = 0$:*

$$(Lu)(x_0) \geq 0. \tag{4.1}$$

L satisfies the quasipositivity principle *if there is a decomposition*

$$L = P - B, \tag{4.2}$$

such that P satisfies the positivity principle and B extends to a bounded linear operator from $C(K)$ to $C(K)$. The nonpositivity index *is given by:*

$$\beta(L) = \inf \|B\|_{\mathcal{L}(C(K))}, \tag{4.3}$$

where $\|\cdot\|_{\mathcal{L}(C(K))}$ is the operator norm from $C(K)$ to $C(K)$ and the infimum is taken over all decomposition of L of the type (4.2)*. If there is a decomposition for which the infimum is attained:*

$$L = P - B,\ \|B\|_{\mathcal{L}(C(K))} = \beta(L), \tag{4.4}$$

we shall call this an optimal decomposition*.*

The positivity principle defined above is referred to as the *positive minimum principle* in [3]. We note that an operator that satisfies the positivity principle trivially satisfies the quasipositivity principle with $B = 0$ in (4.2).

An operator L satisfies the quasipositivity principle if and only if the nonpositivity index $\beta(L) < \infty$. If L satisfies the positivity principle, then $\beta(L) = 0$. The converse is also true, as can be seen as follows. Suppose $u \geq 0$ is in $\mathcal{D}(L)$ and is equal to 0 at x_0. Given that K is compact, u is a bounded function. By (4.3), for any $\epsilon > 0$ there is a decomposition $L = P - B$, such that P satisfies the positivity principle and $\|B\|_{\mathcal{L}(C(K))} \leq \epsilon$. Thus,

$$(Lu)(x_0) = (Pu)(x_0) - (Bu)(x_0) \geq -\|B\|_{\mathcal{L}(C(K))} \|u\|_{C(K)} \geq -\epsilon \|u\|_{C(K)}. \tag{4.5}$$

Since $\epsilon > 0$ is arbitrary, $(Lu)(x_0) \geq 0$. Hence, L satisfies the positivity principle.

We shall henceforth make the following assumption on L for simplicity:

$$1_K \in \mathcal{D}(L), \tag{4.6}$$

where 1_K is the function defined by

$$1_K \equiv 1 \quad \text{on } K. \tag{4.7}$$

We remark that it is not necessary to make this assumption in the ensuing discussion. In fact, all we need is that $\mathcal{D}(L)$ contains a function that is strictly positive on K, and this condition is automatically satisfied since $\mathcal{D}(L)$ is assumed dense in $C(K)$ (see [3]). However, we will not pursue such generality, as adding the assumption (4.6) does not affect the later arguments of this paper.

To discuss semigroup generation of quasipositive operators, we shall make use of the following [3].

Definition 4.2. *Define for $u \in C(K)$ the following functionals p_+ and p_n:*

$$p_+(u) = \sup_{x\in\Gamma} u^+(x), \; u^+(x) = \max(u(x), 0), \tag{4.8}$$

$$p_n(u) = \|u\|_{C(\Gamma)} = \sup_{x\in\Gamma} |u(x)|\,. \tag{4.9}$$

Denote by ∂p the subdifferential of $p = p_+, p_n$:

$$\partial p(u) = \{m \in \mathcal{M}(K) |\, \langle v, m\rangle \le p(v) \text{ for all } v \in C(K), \langle u, m\rangle = p(u)\}, \tag{4.10}$$

where $\mathcal{M}(K)$ is the set of Radon measures on K, and $\langle\cdot,\cdot\rangle$ denotes duality pairing.

*An operator L is p-*dissipative *($p = p_+$ or p_n) if for every $u \in \mathcal{D}(L)$ there exists an $m \in \partial p(u)$ such that $\langle Lu, m\rangle \le 0$.*

The following result is known for the generator of semigroups on $C(K)$.

Proposition 4.3 ([3])**.** *Let L be a densely defined p_+-dissipative operator on $C(K)$. If the range of $\lambda - L$ is dense in $C(K)$ for some $\lambda > 0$, then the closure of L is a generator of a positive contraction semigroup on $C(K)$. If L is p_n-dissipative under the same conditions, it is a generator of a (not necessarily positive) contraction semigroup on $C(K)$.*

This proposition can be seen as a $C(K)$ analogue of the Lumer-Phillips criteria for the generation of contraction semigroups on Hilbert spaces. We first have the following immediate consequence.

Proposition 4.4. *Suppose L satisfies the positivity principle and $1_K \in \mathcal{D}(L)$. Let*

$$\lambda_L = \|L1_K\|_{C(K)}\,. \tag{4.11}$$

Suppose there is a $\lambda > \lambda_L$ such that the image of $\lambda - L$ is dense in $C(K)$. Then the closure of L (which we continue to denote by L) is a generator of a positive semigroup on $C(\Gamma)$ that satisfies the bound:

$$\|\exp(tL)\|_{\mathcal{L}(C(K))} \le \exp(\lambda_L t). \tag{4.12}$$

Proof. We show that $L - \lambda_L$ is p_+-dissipative. By Proposition 4.3, this will imply that $L - \lambda_L$ generates a positive contraction semigroup. From this, we can immediately conclude that L generates a positive semigroup and that (4.12) holds.

Take any function $u \in \mathcal{D}(L)$. What we have to show is that there exists an $m \in \partial p_+(u)$ such that $\langle (L - \lambda_L)u, m\rangle \le 0$. We assume $p_+(u) > 0$ for if $p_+(u) = 0$ then the above claim trivially holds with $m = 0$. Since Γ is compact, there exists a point x_0 such that $u(x_0) = p_+(u) > 0$. The Dirac mass $\delta_{x_0} \in \mathcal{M}(K)$ located at $x = x_0$ clearly belongs to $\partial p_+(u)$. Thus, it suffices to show that

$$\langle (L - \lambda_L)u, \delta_{x_0}\rangle = (Lu)(x_0) - \lambda_L u(x_0) \le 0. \tag{4.13}$$

Since $h \geq 0$, $h(x_0) = 0$, and since L satisfies the positivity principle, $(Lh)(x_0) \geq 0$. This establishes (4.13). □

In the quasipositive case, we have the following.

Proposition 4.5. *Let L satisfy the quasipositivity principle and suppose $1_K \in \mathcal{D}(L)$. Suppose that there is a $\lambda > \mu = \lambda_L + 2\beta(L)$ such that the image of $\lambda - L$ is dense in $C(K)$ (λ_L was defined in (4.11)). Then, the closure of L (which we continue to denote by L) is a generator of a semigroup on $C(K)$ and satisfies the following estimate.*

$$\|\exp(tL)\|_{\mathcal{L}(C(K))} \leq \exp((\lambda_L + 2\beta(L))t), \tag{4.14}$$

where $\beta(L)$ is the nonpositivity index defined in (4.3) *and λ_L is the same as in the previous proposition.*

Proof. We prove that $L - \mu$ is a generator of a contraction semigroup on $C(K)$. Given Proposition 4.3, we must show that $L - \mu$ is p_n-dissipative on $C(K)$.

Take a function $u \in \mathcal{D}(L)$ and suppose the value of the sup norm is attained at $x_0 \in \Gamma$. We assume henceforth that $u(x_0)$ is positive; the case when $u(x_0)$ is negative can be handled analogously. Following the same argument as in the previous proposition, we must show that $((L - \mu)u)(x_0) \leq 0$.

The quasipositivity of L implies that it admits a decomposition $L = P_\delta - B_\delta$ where P_δ satisfies the positivity principle and

$$\|B_\delta\|_{\mathcal{L}(C(K))} \leq \beta(L) + \delta,\ \delta > 0. \tag{4.15}$$

The number δ may be taken arbitrarily small. First note

$$((L - \mu)u)(x_0) = u(x_0)(L1_K)(x_0) - (L(u(x_0)1_K - u))(x_0) - \mu u(x_0). \tag{4.16}$$

Let $h(x) = u(x_0)1_K - u(x) \geq 0$.

$$\begin{aligned}
&u(x_0)(L1_\Gamma)(x_0) - (Lh)(x_0) - \lambda(u)(x_0)\\
\leq&\lambda_L u(x_0) - (P_\delta h)(x_0) + (B_\delta h)(x_0) - \mu u(x_0)\\
\leq&\lambda_L u(x_0) + (B_\delta h)(x_0) - \mu u(x_0)\\
\leq&\lambda_L u(x_0) + \|B_\delta\|_{\mathcal{L}(C(\Gamma))} \|h(x)\|_{C(\Gamma)} - \mu u(x_0) \leq 2\delta u(x_0).
\end{aligned} \tag{4.17}$$

The second inequality follows from the positivity of P. In the fourth inequality we used $\|h(x)\|_{C(K)} \leq 2u(x_0)$ and the definition of μ. Since $\delta > 0$ was arbitrary, we obtain (4.16). □

We shall find the following inequality useful in Section 6.

Corollary 4.6. *Suppose L satisfies the quasipositivity principle and $u \in \mathcal{D}(L)$. Let $\|u\|_{C(K)} = M$ and suppose $u(x_0) = M$.*

$$(Lu)(x_0) \leq \beta(L) \|M - u\|_{C(K)} + \lambda_L M \leq (2\beta(L) + \lambda_L)M. \tag{4.18}$$

Proof. The proof of this inequality is contained in the proof of the previous proposition. □

4.2. **Quasipositivity of $-\Lambda_i$ and $-\Lambda_\sigma$.** Now we turn to the operators $-\Lambda_i$ and $-\Lambda_\sigma$ defined in Section 2. Our goal is to show that $-\Lambda_i$ and $-\Lambda_\sigma$ satisfy the quasipositivity and positivity principles in the space $C(\Gamma)$. Note also that $1_\Gamma = 1_K$ is in the domain of Λ_i and λ_σ. For $-\Lambda_i$, we have the following.

Proposition 4.7. *The operator $-\Lambda_i$ satisfies the positivity principle.*

Proof. This is a simple consequence of the maximum principle for the Laplacian. Let $u \geq 0$ be a smooth function on Γ and $u(x_0) = 0$. Let u_i be the solution to the boundary value problem (2.9). Then,

$$(\Lambda_i u)(x_0) = \left.\frac{\partial u_i}{\partial \mathbf{n}}\right|_{x=x_0}, \tag{4.19}$$

where $\mathbf{n}$ is the outward unit normal on Γ. By the maximum principle, u_i must attain its minimum value at x_0. Therefore, the normal derivative $\frac{\partial u_i}{\partial \mathbf{n}}$ must be non-positive at $x = x_0$. □

We now turn to $-\Lambda_\sigma$. For illustrative purposes, we shall first focus on the easier case $\sigma = 1$. In this case, the solution of the boundary value problem (2.5a)-(2.5c) can be written directly in terms of the double layer potential. Given a smooth u on Γ, the action of Λ_1 can be written as:

$$(\Lambda_1 u)(x) = \frac{\partial}{\partial \mathbf{n}_x} \int_\Gamma D(x,y) u(y) dS_y, \tag{4.20}$$

$$D(x,y) = -\frac{(x-y)\cdot \mathbf{n}_y}{4\pi |x-y|^3}, \tag{4.21}$$

where $\mathbf{n}_x$ is the unit outward normal on Γ at x. The positivity of $-\Lambda_1$ is linked to the negativity of the following kernel:

$$K_1(x,y) \equiv \frac{\partial}{\partial \mathbf{n}_x} D(x,y), \ x \neq y. \tag{4.22}$$

Let

$$K_1^B(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \max(K_1(x,y), 0) & \text{if } x \neq y. \end{cases} \tag{4.23}$$

Define

$$B_1 u = \int_\Gamma K_1^B(x,y) u(y) dS_y, \ P_1 u = -\Lambda_1 u + B_1 u. \tag{4.24}$$

Proposition 4.8. $-\Lambda_1$ *satisfies the quasipositivity principle. Moreover,* (4.24) *is an optimal decomposition. In particular, if $K_1(x,y) \leq 0$ then $-\Lambda_1$ satisfies the positivity principle.*

Proof. Let us write down $K_1(x,y)$:

$$K_1(x,y) = \frac{1}{4\pi |x-y|^3} \left(-\mathbf{n}_x \cdot \mathbf{n}_y + 3(\mathbf{n}_x \cdot \hat{\mathbf{r}})(\mathbf{n}_y \cdot \hat{\mathbf{r}})\right) \leq 0, \ \hat{\mathbf{r}} = \frac{x-y}{|x-y|}. \tag{4.25}$$

Given that Γ is smooth, we see that $K_1(x,y)$ is negative whenever $|x-y|$ is small enough. Since Γ is compact, there is a real number $\rho > 0$ such that $K(x,y) \leq 0$ whenever $|x-y| \leq \rho$. It is clear from this observation that $K_1^B(x,y)$ is a bounded continuous function on $\Gamma \times \Gamma$ and thus B is a bounded operator acting on $C(\Gamma)$.

We now show that P is a positive operator. Let $u \geq 0$, $u \in C^\infty(\Gamma)$ and suppose $u(x_0) = 0$.

$$\begin{aligned}(P_1 u)(x_0) &= -\frac{\partial}{\partial \mathbf{n}_{x_0}} \int_\Gamma D(x_0, y)u(y)dS_y + \int_\Gamma K_1^B(x_0, y)u(y)dS_y \\ &= I_1 + I_2, \\ I_1 &= -\frac{\partial}{\partial \mathbf{n}_{x_0}} \int_{\Gamma \cap |y-x_0|>\epsilon} D(x_0, y)u(y)dS_y + \int_\Gamma K_1^B(x_0, y)u(y)dS_y, \\ I_2 &= -\frac{\partial}{\partial \mathbf{n}_{x_0}} \int_{\Gamma \cap |y-x_0|\leq\epsilon} D(x_0, y)u(y)dS_y.\end{aligned} \tag{4.26}$$

Let $\epsilon < \rho$. Consider I_1.

$$\begin{aligned}I_1 &= -\frac{\partial}{\partial \mathbf{n}_{x_0}} \int_{\Gamma \cap |y-x_0|>\epsilon} D(x_0, y)u(y)dS_y + \int_\Gamma K_1^B(x_0, y)u(y)dS_y \\ &= \int_{\Gamma \cap |y-x_0|>\epsilon} (-K_1(x, y) + K_1^B(x, y))u(y)dS_y.\end{aligned} \tag{4.27}$$

By (4.23), $-K_1(x, y) + K_1^B(x, y)$ is non-negative for $x \neq y$ and since $u \geq 0$, $I_1 \geq 0$. We now turn to I_2. We may rewrite (4.22) as: Since u attains a minimum at x_0 and is smooth, for ϵ small enough there is a constant C such that

$$|u(y)| \leq C\,|y - x_0|^2 \text{ for } |y - x_0| \leq \epsilon. \tag{4.28}$$

Thus, for $|y - x_0| \leq \epsilon$,

$$\left|\frac{\partial}{\partial \mathbf{n}_{x_0}} D(x_0, y)u(y)\right| = |K_1(x_0, y)u(y)| \leq \frac{C}{|y - x_0|}, \tag{4.29}$$

for some constant C. Since the last expression in the above is integrable over a two-dimensional smooth surface,

$$\begin{aligned}|I_2| &\leq \int_{\Gamma \cap |y-x_0|} |K_1(x_0, y)h(y)|\,dS_y \\ &\leq \int_{\Gamma \cap |y-x_0|} \frac{C}{|y - x_0|} dS_y \leq (2\pi + 1)C\epsilon.\end{aligned} \tag{4.30}$$

Thus,

$$(P_1 u)(x_0) = I_1 + I_2 \geq -(2\pi + 1)C\epsilon. \tag{4.31}$$

Since ϵ can be made arbitrarily small, $(Pu)(x_0) \geq 0$. This shows that $-\Lambda_1$ satisfies the quasipositivity principle.

To show that $\beta(-\Lambda_1) = \|B\|_{\mathcal{L}(C(\Gamma))}$, consider an arbitrary decomposition $-\Lambda_1 = \tilde{P}_1 - \tilde{B}_1$ such that $\tilde{P}_1$ satisfies the positivity principle and $\tilde{B}_1 \in \mathcal{L}(C(\Gamma))$. Fix a point $x \in \Gamma$, and consider the set

$$\Gamma_x^+ = \{y \in \Gamma | K_1(x, y) > 0\}. \tag{4.32}$$

Let $C_c^\infty(\Gamma_x^+)$ denote the set of smooth functions compactly supported on Γ_x^+. Take any $u \in C_c^\infty(\Gamma_x^+)$. We have

$$(\tilde{B}_1 u)(x) = (\tilde{P}_1 u)(x) + (\Lambda_1 u)(x) = (\tilde{P}_1 u)(x) + (B_1 u)(x) - (P_1 u)(x). \tag{4.33}$$

Note that $u(x) = 0$, and therefore $(\tilde{P}_1 u)(x) \geq 0$ since $\tilde{P}_1$ satisfies the positivity principle. Since $u \in C_c^\infty(\Gamma_x^+)$, $P_1 u = 0$. Thus, we see that

$$(\tilde{B}_1 u)(x) \geq (B_1 u)(x). \tag{4.34}$$

Let $\mathcal{D}$ be the set of positive functions of $C_c^\infty(\Gamma_x^+)$ whose norm in $C(\Gamma)$ does not exceed 1. We have:

$$\begin{aligned}\left\|\tilde{B}_1\right\|_{\mathcal{L}(C(\Gamma))} &= \sup_{x\in\Gamma}\sup_{\|u\|_{C(\Gamma)}\leq 1}\left|(\tilde{B}_1u)(x)\right| \\ &\geq \sup_{x\in\Gamma}\sup_{u\in\mathcal{D}}(\tilde{B}_1u)(x) \geq \sup_{x\in\Gamma}\sup_{u\in\mathcal{D}}(B_1u)(x) \\ &= \sup_{x\in\Gamma}\int_\Gamma K_1^B(x,y)dS_y = \|B_1\|_{\mathcal{L}(C(\Gamma))}.\end{aligned} \tag{4.35}$$

where we used (4.34) in the second inequality. □

Given a specific geometry for Γ, we can determine the positivity of Λ_1 by examining the sign of the following expression in the parentheses of equation (4.25):

$$I_{xy} = -\mathbf{n}_x\cdot\mathbf{n}_y + 3(\mathbf{n}_x\cdot\hat{\mathbf{r}})(\mathbf{n}_y\cdot\hat{\mathbf{r}}). \tag{4.36}$$

If the above quantity is non-positive for all $\mathbf{x},\mathbf{y}\in\Gamma$, then $\beta(-\Lambda_1)=0$ and $-\Lambda_1$ is a positive operator. Let us compute this quantity for a few different geometries.

When Γ is a sphere, (4.36) may be computed as:

$$I_{xy} = -\frac{1}{2}(3-\mathbf{n}_x\cdot\mathbf{n}_y) < 0. \tag{4.37}$$

Therefore, $-\Lambda_1$ is a positive operator when Γ is a sphere.

In general, however, $\beta(-\Lambda_1)$ is not necessarily equal to 0. A very easy example is the torus:

$$(\sqrt{x_1^2+x_2^2}-R)^2+z^2=r^2,\ R>r>0. \tag{4.38}$$

Take the two points:

$$\mathbf{x}=(R-r,0,0),\ \mathbf{y}=(r-R,0,0). \tag{4.39}$$

Then, $\mathbf{n}_x=-\mathbf{n}_y$ and $\hat{\mathbf{r}}=-\mathbf{n}_x=\mathbf{n}_y$, and therefore,

$$I_{xy}=4. \tag{4.40}$$

It is quite easy to see from this example that $-\Lambda_1$ is not positive if the membrane Γ "folds upon itself".

One might be led to think that $-\Lambda_1$ is positive if the domain Ω_i is convex. This is, in fact, not the case. The convexity of Ω_i does imply that the second term in (4.36), $3(\mathbf{n}_x\cdot\hat{\mathbf{r}})(\mathbf{n}_y\cdot\hat{\mathbf{r}})$, must be negative. Therefore, $I_{xy}<1$ for convex domains, but we can achieve any value arbitrarily close to 1, as we shall now see.

Let Γ be the following ellipsoid:

$$\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}=0,\ a_1\geq a_2\geq a_3>0. \tag{4.41}$$

In Appendix A, we prove that $-\Lambda_1$ is positive if and only if:

$$\frac{a_1}{a_3}\leq\sqrt{2}+\sqrt{3}. \tag{4.42}$$

In the case of the ellipsoid, $-\Lambda_1$ is positive only if it is sufficiently "close" to a sphere.

The quasipositivity of $-\Lambda_1$ has the following physical interpretation (See Fig. 2). We see from (4.25) that $K_1(x,y)$ can be written as:

$$K_1(x,y)=\mathbf{F}_D(x,y)\cdot\mathbf{n}_x,\ \mathbf{F}_D=\frac{1}{4\pi|x-y|^3}\left(-\mathbf{n}_y+3(\mathbf{n}_\mathbf{y}\cdot\hat{\mathbf{r}})\mathbf{n}_y\right), \tag{4.43}$$

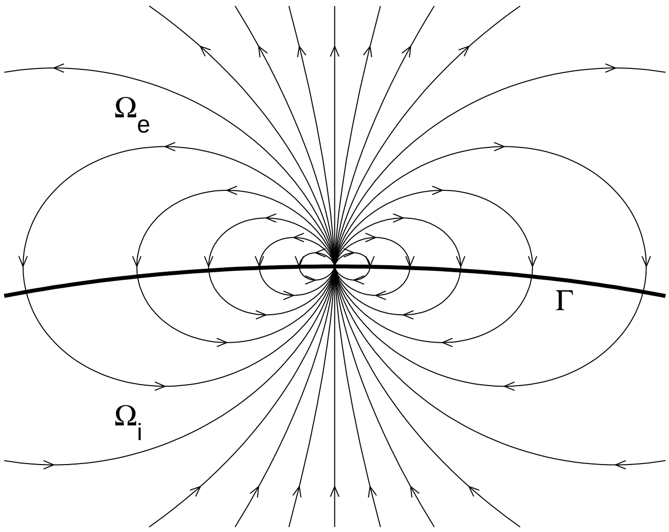

FIGURE 2. The electric field generated by a dipole placed on the membrane whose moment is the outward normal. Notice that, in the vicinity of the dipole, the field lines penetrate the membrane Γ from the Ω_e side to the Ω_i side of the membrane.

where $\mathbf{F}_D(x,y)$ may be interpreted as the electric field at $x \in \Gamma$ generated by a dipole at $y \in \Gamma$ with moment $\mathbf{n}_y$. Quasipositivity of $-\Lambda_1$ follows from the fact that $\mathbf{F}_D \cdot \mathbf{n}_x$ is negative whenever x is close to y. In other words, the electric field lines of $\mathbf{F}_D$ always penetrates the membrane Γ from Ω_e to Ω_i when x is sufficiently close to y (Fig. 2). In light of this interpretation, we see that $-\Lambda_1$ is positive if the electric field lines penetrate the membrane in the Ω_e to Ω_i direction for any $x, y \in \Gamma$. This point is explained in Fig. 3 in the case when Γ is a sphere or an ellipsoid.

We now turn to the general case.

Proposition 4.9. $-\Lambda_\sigma, 0 < \sigma < \infty$ *satisfies the quasipositivity principle. Moreover,* $-\Lambda_\sigma$ *admits an optimal decomposition.*

Proof. We shall first treat the case $1 \leq \sigma < \infty$. Recall from (3.18) that Λ_σ can be written as:

$$\Lambda_\sigma u = \Lambda_\infty u - \frac{1}{2}(1-\epsilon)(1+2T_S)(1-2\epsilon T_S)^{-1}\Lambda_\infty u, \tag{4.44}$$

for $u \in C^\infty(\Gamma)$, where $\epsilon = \frac{\sigma-1}{\sigma+1}$. In particular,

$$\Lambda_1 u = \left(\frac{1}{2} - T_S\right)\Lambda_\infty u. \tag{4.45}$$

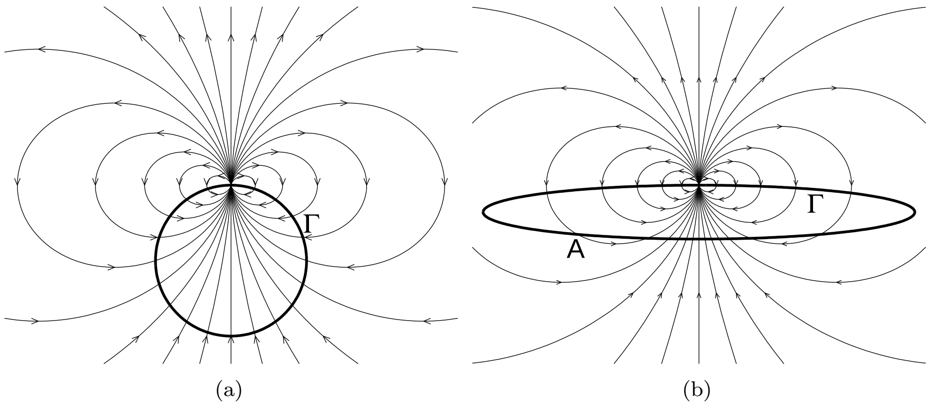

FIGURE 3. (a): If Γ is a sphere, the electric field lines of the dipole penetrates the membrane from the outside to the inside of the cell at all points on Γ. The operator $-\Lambda_1$ is positive in this case. (b): If Γ is an ellipsoid, the electric field lines may penetrate the membrane from the inside to the outside of the cell. This is seen to happen at point A in the figure. In this case, $-\Lambda_1$ is not positive.

Let us now calculate:

$$\begin{aligned}
&\Lambda_\sigma u - (1+\epsilon)\left(\frac{\epsilon}{2}\Lambda_\infty + (1-\epsilon)\Lambda_1\right)u \\
=&\Lambda_\infty u - \frac{1}{2}(1-\epsilon)(1+2T_S)(1-2\epsilon T_S)^{-1}\Lambda_\infty u \\
&-(1+\epsilon)\left(\frac{\epsilon}{2}\Lambda_\infty + (1-\epsilon)\left(\frac{1}{2}-T_S\right)\Lambda_\infty\right)u \\
=&\frac{1}{2}(1-\epsilon)(1+2T_S+2\epsilon T_S-(1+2T_S)(1-2\epsilon T_S)^{-1})\Lambda_\infty u \\
=&-2\epsilon(1-\epsilon^2)(1-2\epsilon T_S)^{-1}T_S^2\Lambda_\infty u \equiv \mathcal{R}_\sigma u.
\end{aligned} \tag{4.46}$$

Note that $\mathcal{R}_\sigma \in \Psi^{-1}(\Gamma)$ since $T_S \in \Psi^{-1}(\Gamma)$ and $\Lambda_\infty \in \Psi^1(\Gamma)$. Thus, $\mathcal{R}_\sigma$ has an integral representation of the form:

$$(\mathcal{R}_\sigma u)(x) = \int_\Gamma K_\sigma^{\mathcal{R}}(x,y)u(y)dS_y, \tag{4.47}$$

where the kernel $K_\sigma^{\mathcal{R}}$ satisfies the bound:

$$\left|K_\sigma^{\mathcal{R}}(x,y)\right| \le \frac{C}{|x-y|}, \tag{4.48}$$

for some constant C that does not depend on x or y. Since $K_\sigma^{\mathcal{R}}(x,y)$ is integrable in y whose value is bounded uniformly in x, $\mathcal{R}_\sigma$ is a bounded operator from $C(\Gamma)$ to $C(\Gamma)$. By Proposition 4.8, Λ_1 can be written as $P_1 - B_1$ where P_1 satisfies the positivity principle and $B_1 \in \mathcal{L}(C(\Gamma))$. Equation (4.46) shows that

$$\begin{aligned} -\Lambda_\sigma u &= \tilde{P}_\sigma u - \tilde{B}_\sigma u, \\ \tilde{P}_\sigma &= \frac{\epsilon(1+\epsilon)}{2}(-\Lambda_\infty) + (1-\epsilon^2)P_1, \\ \tilde{B}_\sigma &= (1-\epsilon^2)B_1 + \mathcal{R}_\sigma. \end{aligned} \tag{4.49}$$

Since $1 \leq \sigma < \infty$, $0 \leq \epsilon < 1$ and thus $\tilde{P}$ is a positive linear combination of two operators that satisfy the positivity principle. Thus, $\tilde{P}_\sigma$ also satisfies the positivity principle. It is clear that $\tilde{B}_\sigma \in \mathcal{L}(C(\Gamma))$. This shows that $-\Lambda_\sigma$ satisfies the quasipositivity principle.

We now construct an optimal decomposition. This construction gives another proof of quasipositivity of $-\Lambda_\sigma$. Let $G(x,y)$ be the Green's function for the homogeneous Dirichlet problem for the domain Ω_i. Then, Λ_∞ can be written as:

$$(\Lambda_\infty u)(x) = \frac{\partial}{\partial \mathbf{n}_x}\int_\Gamma \frac{\partial}{\partial \mathbf{n}_y} G(x,y)u(y)dS_y. \tag{4.50}$$

Therefore, (4.46) implies that Λ_σ has the following integral representation:

$$\begin{aligned} (\Lambda_\sigma)u = &\frac{1}{2}\epsilon(1+\epsilon)\frac{\partial}{\partial \mathbf{n}_x}\int_\Gamma \frac{\partial}{\partial \mathbf{n}_y} G(x,y)u(y)dS_y \\ &+ (1-\epsilon^2)\frac{\partial}{\partial \mathbf{n}_x}\int_\Gamma D(x,y)u(y)dS_y \\ &+ \int_\Gamma K_\sigma^{\mathcal{R}}(x,y)u(y)dS_y. \end{aligned} \tag{4.51}$$

Consider the kernel K_σ defined for $x \neq y$.

$$K_\sigma(x,y) = \frac{1}{2}\epsilon(1+\epsilon)K_\infty(x,y) + (1-\epsilon^2)K_1(x,y) + K_\sigma^{\mathcal{R}}(x,y), \tag{4.52}$$

where $K_\infty(x,y) = \frac{\partial}{\partial \mathbf{n}_x}\left(\frac{\partial}{\partial \mathbf{n}_y}G(x,y)\right)$. Note that $K_\sigma(x,y)$ is a smooth function. Formula (4.25) implies that $K_1(x,y) = \frac{\partial}{\partial \mathbf{n}_x}D(x,y)$ goes like $-\frac{1}{|x-y|^3}$ when $|x-y|$ is small. By (4.48), we see that there is a real number $\rho > 0$ such that

$$(1-\epsilon^2)K_1(x,y) + K_\sigma^{\mathcal{R}}(x,y) \leq 0 \tag{4.53}$$

for any x, y such that $|x-y| \leq \rho$. Since $-\Lambda_\infty$ satisfies the positivity principle, it follows that $K_\infty(x,y) \leq 0$ for all $x \neq y$. This implies that

$$K_\sigma(x,y) \leq 0 \text{ for } |x-y| \leq \rho. \tag{4.54}$$

Let

$$K_\sigma^B(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \max(K_\sigma(x,y), 0) & \text{if } x \neq y. \end{cases} \tag{4.55}$$

Since $K_\sigma(x,y)$ is a smooth function and is thus bounded when $|x-y| \geq \rho$, $K_\sigma^B(x,y)$ is a bounded continuous function. Define

$$(B_\sigma u)(x) = \int_\Gamma K_\sigma^B u(y)dS_y. \tag{4.56}$$

We see that $B_\sigma \in \mathcal{L}(C(\Gamma))$. Consider a smooth function $u(x) \geq 0$ such that $u(x_0) = 0$. Let ψ be a smooth non-negative cutoff function defined on Γ such that

$$\psi(x) = \begin{cases} 1 & \text{when } |x - x_0| \leq \rho/2, \\ 0 & \text{when } |x - x_0| \geq \rho. \end{cases} \tag{4.57}$$

Let $P_\sigma = -\Lambda_\sigma + B_\sigma$.

$$(P_\sigma u)(x_0) = (P_\sigma(\psi u))(x_0) + (P_\sigma((1-\psi)u))(x_0). \tag{4.58}$$

The first term can be written as:

$$\begin{aligned} (P_\sigma(\psi u))(x_0) &= J_1 + J_2, \\ J_1 &= -\frac{1}{2}\epsilon(1+\epsilon)(\Lambda_\infty(\psi u))(x_0), \\ J_2 &= -(1-\epsilon^2)(\Lambda_1(\psi u))(x_0) + (\mathcal{R}_\sigma(\psi u))(x_0). \end{aligned} \tag{4.59}$$

$J_1 \geq 0$ since $-\Lambda_\infty$ satisfies the positivity principle. $J_2 \geq 0$ can be shown using an argument analogous to Proposition 4.8. The second term of (4.58) can be written as:

$$(P_\sigma((1-\psi)u))(x_0) = \int_\Gamma \max(-K_\sigma(x_0, y), 0)(1-\psi(y))u(y)dS_y. \tag{4.60}$$

This is clearly non-negative. That this is an optimal decomposition follows from an argument analogous to Proposition 4.8.

We now turn to the $0 < \sigma \leq 1$ case. Consider the (2.5a)-(2.5c) where (2.5b) is replaced by:

$$\frac{\partial u_e}{\partial \boldsymbol{\nu}} = \tau \frac{\partial u_i}{\partial \boldsymbol{\nu}}, \; u_e - u_i = u \text{ on } \Gamma, \tag{4.61}$$

where $\boldsymbol{\nu} = -\mathbf{n}$ is the inward pointing normal on Γ and $\tau > 0$. Note how the roles of Ω_e and Ω_i are reversed. Introduce the following operator associated with this problem.

$$\tilde{\Lambda}_\tau u = \frac{\partial u_e}{\partial \boldsymbol{\nu}}. \tag{4.62}$$

The above is the same problem as (2.5a)-(2.5c) if we take $\tau = 1/\sigma$ and replace u by $-u$ in (4.61). Therefore,

$$\begin{aligned} \tilde{\Lambda}_\tau u = \frac{\partial u_e}{\partial \nu} &= -\frac{\partial u_e}{\partial \mathbf{n}} = -\tau \frac{\partial u_i}{\partial \mathbf{n}} \\ &= -\tau \Lambda_{1/\tau}(-u) = \tau \Lambda_{1/\tau} u. \end{aligned} \tag{4.63}$$

We see that Λ_σ satisfying the quasipositivity principle for $0 < \sigma \leq 1$ is equivalent to $\tilde{\Lambda}_\tau$ satisfying the quasipositivity principle for $1 \leq \tau < \infty$. We thus seek to decompose $\tilde{\Lambda}_\tau$ into a positive and bounded part:

$$-\tilde{\Lambda}_\tau = \tilde{P}_\tau - \tilde{B}_\tau. \tag{4.64}$$

We let $\tilde{\Lambda}_\infty$ be the Dirichlet-Neumann map for the exterior Dirichlet problem. Consider

$$\Delta u_e = 0 \text{ in } \Omega_e, \tag{4.65}$$

$$u_e = u \text{ on } \Gamma, \; u_e(x) \to 0 \text{ as } |x| \to \infty. \tag{4.66}$$

We set:

$$\tilde{\Lambda}_e = \tilde{\Lambda}_\infty = \frac{\partial u_e}{\partial \boldsymbol{\nu}}. \tag{4.67}$$

The operator $\tilde{\Lambda}_\infty$ satisfies the positivity principle as a direct consequence of the maximum principle for the Laplace equation, and it is a pseudodifferential operator of order 1. We can thus follow the same computation used for $\Lambda_\sigma, 1 \leq \sigma < \infty$ to conclude that $\tilde{\Lambda}_\tau, 1 \leq \tau < \infty$ satisfies the approximate positivity principle. The optimal decomposition can be constructed in a similar fashion. □

4.3. **Semigroup generation in $C(\Gamma)$ and $W^s_p(\Gamma)$.** Now that we have established the positivity and quasipositivity principles for $-\Lambda$ with $\Lambda = \Lambda_i$ or Λ_σ, we may use results from Section 4.1 to obtain semigroup generation results. Our strategy is the following:

(a) Recall the analyticity of $\exp(-t\Lambda)$ in $L^2(\Gamma)$ (or $H^s(\Gamma)$) which is a consequence of the self-adjointness of Λ (Proposition 3.1).
(b) Prove that $\exp(-t\Lambda)$ is a C^0 semigroup on $C(\Gamma)$, which is a consequence of the quasi-positivity of $-\Lambda$ (see Proposition 4.10 below).
(c) Apply a general interpolation argument to show the analyticity of $\exp(-t\Lambda)$ on $W^s_p(\Gamma), 1 < p < \infty$ (see Proposition 4.13 below).

We begin with step (b). In the proof of the statements to follow, we shall often make use of the following fact.

$$\Lambda 1_\Gamma = 0. \tag{4.68}$$

This can be easily seen by solving the (2.5a)-(2.5c) or (2.9).

Proposition 4.10. *Let $\Lambda = \Lambda_i$ or Λ_σ. Then, $-\Lambda$ generates a semigroup* $\exp(-t\Lambda)$ *on $C(\Gamma)$. If $-\Lambda$ satisfies the positivity principle, then* $\exp(-t\Lambda)$ *is a positive contraction semigroup.*

Proof. Take the domain of Λ to be the set of smooth functions $C^\infty(\Gamma)$. We apply Proposition 4.5. Given (4.68), $\lambda_L = 0$, where λ_L was defined in (4.11). Therefore, it suffices to show that $\lambda + \Lambda$ maps $C^\infty(\Gamma)$ to a dense subset of $C(\Gamma)$ for some $\lambda > 2\beta(-\Lambda)$. By the proof of Proposition 3.1, $\lambda + \Lambda$ extends to a bounded linear bijection from $H^{s+1}(\Gamma)$ onto $H^s(\Gamma)$ for $s \leq 0, \lambda > 0$. Therefore, $\lambda + \Lambda$ maps $C^\infty(\Gamma)$ onto $C^\infty(\Gamma)$, which is dense in $C(\Gamma)$. If $-\Lambda$ satisfies the positivity principle, a similar argument using Proposition 4.4 leads to the desired conclusion. □

Now, we proceed to step (c). We first prove Proposition 4.11, which is is a variation of a well-known general interpolation result.

Let K be a compact Hausdorff space endowed with a Radon measure ν with the following property.

$$\text{For any nonempty open set } \mathcal{O} \subset K, \ \nu(\mathcal{O}) > 0. \tag{4.69}$$

Take any function $u \in C(K)$. The above property ensures that $\|u\|_{C(K)} = \|u\|_{L^\infty(K,\nu)}$ as can be seen as follows. If $r > \|u\|_{C(K)}$, then the set

$$A_r = \{x \in K \,|\, |u(x)| > r\} \tag{4.70}$$

is empty. Thus, $\nu(A_r) = 0$ and therefore, $\|u\|_{C(K)} \geq \|u\|_{L^\infty(K,\nu)}$. If $r < \|u\|_{C(K)}$, then A_r is a non-empty open set by the continuity of u, and therefore, $\nu(A_r) > 0$. Thus, $\|u\|_{C(K)} \leq \|u\|_{L^\infty(K,\nu)}$.

Proposition 4.11. *Let $T_C(t) = \exp(tL)$ be a C_0 semigroup in $C(K)$ whose generator is L. Suppose that $T_C(t)$ has a (unique) extension in $L^2(K,\nu)$, which we denote by $T_2(t)$, and that $T_2(t)$ is an analytic semigroup on $L^2(K,\nu)$. Here, the Radon measure ν satisfies property* (4.69). *Then, $T_C(t)$ has a unique extension to*

$L^p(K,\nu), 2 < p < \infty$, which we denote by $T_p(t)$, and $T_p(t)$ is an analytic semigroup in $L^p(K,\nu)$.

Proof. Note first that if an extension of $T_C(t)$ to $L^p(K,\nu), 2 \le p < \infty$ exists this should be unique due to the density of $C(K)$ in $L^p(K,\nu)$.

Take a function $u \in L^\infty(K,\nu)$. Modify u on a set of measure 0 so that $|u(x)| \le \|u\|_{L^\infty(K,\nu)}, x \in K$. There is a sequence $u_n \in C(K), \|u_n\|_{C(K)} \le \|u\|_{L^\infty(K,\nu)}, n = 1, 2, \cdots$ such that $u_n(x) \to u(x), a.e. x \in K$ (see for example, [31]).

Given that $\nu(K) < \infty$, $u \in L^2(K,\nu)$ and by the Lebesgue dominated convergence theorem, $u_n \to u$ in $L^2(K,\nu)$. Therefore, $T_2(t)u_n \to T_2(t)u$ in $L^2(K,\nu)$. Take a subsequence of u_n, which we shall still denote by u_n, so that $(T_2(t)u_n)(x) \to (T_2(t)u)(x), a.e. x \in K$. For $x \in K$, we have:

$$\begin{aligned} |(T_2(t)u_n)(x)| = |(T_C(t)u_n)(x)| &\le C_t \|u_n\|_{C(K)} \\ &= C_t \|u_n\| L^\infty(K,\nu) \le C_t \|u\|_{L^\infty(K,\nu)}, \end{aligned} \tag{4.71}$$

where C_t is a constant that depends on t. In the first equality, we used the fact that $T_C(t)$ and $T_2(t)$ coincide on $C(K)$ (since $C(K) \subset L^2(K,\nu)$). In the second equality, we used the equality of $C(K)$ and $L^\infty(K,\nu)$ norms for functions in $C(K)$, and thus property (4.69). Choosing x in (4.71) so that $u_n(x) \to u(x)$ and taking the limit as $n \to \infty$, we have:

$$\|T_2(t)u\|_{L^\infty(K,\nu)} \le C_t \|u\|_{L^\infty(K,\nu)}. \tag{4.72}$$

The restriction of $T_2(t)$ to $L^\infty(K,\nu)$ is thus bounded as a map from $L^\infty(K,\nu)$ to itself.

The remainder of the proof rests on a classical interpolation argument. The details can be found, for example, in Proposition 3.12 of [26] or Theorem 1.4.2 of [5]. □

For $L^p(K,\nu), 2 \le p < \infty$, consider its dual, $L^q(K,\nu), 1/p + 1/q = 1$. By a duality argument, it can be seen that the adjoint semigroup $(T_p(t))^*$ is an analytic semigroup on $L^q(K,\nu)$. Suppose $T_2(t)$ is a symmetric operator on $L^2(K,\nu)$, or equivalently, that the generator L_2 of $T_2(t)$ is self-adjoint. This symmetry will imply that $(T_p(t))^*$ is a (unique) extension of $T_p(t)$ to $L^q(K,\nu)$. Therefore, we have:

Corollary 4.12. *Assume in addition to the hypotheses of Proposition 4.11 that the generator L_2 of $T_2(t)$ is self-adjoint. Then, $T_C(t)$ has a unique extension $T_p(t)$ to $L^p(K,\nu), 1 < p < \infty$ so that $T_p(t)$ is an analytic semigroup on $L^p(K,\nu)$.*

We are now ready to prove analyticity of the $\exp(-t\Lambda)$ in L^p-based sobolev spaces.

Proposition 4.13. *Let $\Lambda = \Lambda_i, \Lambda_\sigma$. The operator $-\Lambda$ generates an analytic semigroup on $W_p^s(\Gamma), s \ge 0, 1 < p < \infty$. The domain of $-\Lambda$ as a semigroup generator in $W_p^s(\Gamma)$ is $W_p^{s+1}(\Gamma)$.*

Proof. From Proposition 3.1 and Proposition 4.10, $\exp(-t\Lambda)$ generates an analytic semigroup on $L^2(\Gamma)$ and a semigroup on $C(\Gamma)$. Given that the surface measure on Γ satisfies property (4.69), $\exp(-t\Lambda)$ satisfies the assumptions of Corollary 4.12. Therefore, $\exp(-t\Lambda)$ defines an analytic semigroup on $L^p(\Gamma), 1 < p < \infty$. Fix a $1 < p < \infty$ and let $\mathcal{D}(\Lambda^k)$ denote the domain of Λ^k in $L^p(\Gamma)$. It is a standard fact that $-\Lambda$ generates a semigroup on $\mathcal{D}(\Lambda^k)$ [19]. Given that $-\Lambda$ generates a semigroup on $L^p(\Gamma)$, $\lambda + \Lambda$ is invertible if λ is taken large enough. Fix such a λ. The domain $\mathcal{D}(\Lambda^k)$ clearly coincides with the image of $L^p(\Gamma)$ under $(\lambda + \Lambda)^{-k}$. Since Λ is an elliptic ΨDO of order 1, the image of $W_p^k(\Gamma), 1 < p < \infty, k \geq 0, k \in \mathbb{Z}$ by $(\lambda + \Lambda)^{-1}$ is $W_p^{k+1}(\Gamma)$. Therefore, $-\Lambda$ generates an analytic semigroup on $W_p^k(\Gamma), k \geq 0, k \in \mathbb{Z}$ and its domain is given by $W_p^{k+1}(\Gamma)$. Note that $W_p^s(\Gamma), s \geq 0, s \notin \mathbb{Z}$ can be realized by real interpolation of the spaces $W_p^k(\Gamma)$ and $W^{k+1}(\Gamma)$ where $k < s < k+1, k \in \mathbb{Z}$ [1]. It is a standard fact that $-\Lambda$ generates analytic semigroups on such interpolation spaces [19]. The corresponding domain of $-\Lambda$ is given by $W_p^{s+1}(\Gamma)$. □

We note that, as far as the one-phase problem ($\Lambda = \Lambda_i$) is concerned, much is known about $\exp(-t\Lambda_i)$. More precisely, [18] establishes analyticity in $H^s(\Gamma)$, and this result is generalized to Besov spaces $B_{pp}^s(\Gamma)$ in [12, 7]. Note that, even for $\Lambda = \Lambda_i$, our results are slightly different in that $B_{pp}^s(\Gamma) \neq W_p^s(\Gamma)$ for $s \in \mathbb{Z}$ (see [1]).

The following proposition gives a slightly refined growth estimate of $\|\exp(-t\Lambda)\|$.

Proposition 4.14. *Let $\Lambda = \Lambda_i, \Lambda_\sigma$. The semigroup $\exp(-t\Lambda)$ satisfies the following bounds for all $t > 0$.*

$$\|\exp(-t\Lambda)u\|_{W_p^s(\Gamma)} \leq M_p^{s,s'}(1 + t^{-(s-s')}) \|u\|_{W_p^{s'}(\Gamma)}, \tag{4.73}$$

$$\|\exp(-t\Lambda)u\|_{C(\Gamma)} \leq M_c \|u\|_{C(\Gamma)}, \tag{4.74}$$

where $0 \leq s' < s < s'+1, s \notin \mathbb{Z}$, the constant $M_p^{s,s'}$ only depends on s, s', p and not on t, and the constant M_c does not depend on t.

Estimate (4.73) holds for $s \in \mathbb{Z}$ if we replace $W_p^s(\Gamma)$ by the Besov space $B_{pp}^s(\Gamma)$. Note that $W_p^s(\Gamma) = B_{pp}^s(\Gamma)$ if $s \notin \mathbb{Z}$.

Proof. By Proposition 3.1 Λ is a positive semidefinite self-adjoint operator with compact resolvent, and thus, the spectrum of Λ on $L^2(\Gamma)$ consist only of eigenvalues that are non-negative and semisimple. The corresponding eigenfunctions form an orthogonal basis of $L^2(\Gamma)$. We note in particular that 0 is a semisimple eigenvalue of Λ in L^2.

Let us now consider the spectrum of Λ seen as an operator on $W_p^s(\Gamma), s \geq 0, 1 < p < \infty$. As we saw in the proof of the previous proposition, $(\lambda + \Lambda)^{-1}$, for λ large, maps $W_p^s(\Gamma)$ to $W_p^{s+1}(\Gamma)$ and is thus compact. This implies that the spectrum of Λ consists only of eigenvalues. Take an eigenvalue ν of Λ:

$$\Lambda u = \nu u, \tag{4.75}$$

where equality is in $W_p^s(\Gamma), 1 < p < \infty$. This immediately implies that $u \in W_p^{s+1}(\Gamma)$, the domain of Λ. Since Λ is an elliptic ΨDO of order 1, this implies that $u \in W_p^{s+2}(\Gamma)$ by elliptic regularity. A bootstrapping argument shows that u is in fact smooth. We thus see that an eigenfunction for a certain W_p^s space must necessarily be an eigenfunction with the same eigenvalue for all other W_p^s spaces. The same argument can be made for generalized eigenfunctions, and thus, the semisimplicity of the eigenvalues is also the same regardless of for all W_p^s spaces. Thus, the spectral information obtained in L^2 carries over to all other W_p^s spaces. In particular, the eigenvalue 0 is semisimple when Λ is considered an operator on $W_p^s(\Gamma), 1 < p < \infty$. Now, (4.73) follows since $\exp(-t\Lambda)$ is an analytic semigroup and $W_p^{s'}(\Gamma)$ is a real interpolation space between $W_p^s(\Gamma)$ and $W_p^{s+1}(\Gamma)$ provided $s' \notin \mathbb{Z}$. (see Section 2.3.1 of [19]).

We now turn to (4.74). Using Proposition 4.5 and (4.68), we have:

$$\|\exp(-t\Lambda)\|_{\mathcal{L}(C(\Gamma))} \leq \exp(\mu t), \tag{4.76}$$

where $\mu = 2\beta(-\Lambda)$. To conclude that the norm $\|\exp(-t\Lambda)\|_{\mathcal{L}(C(\Gamma))}$ is uniformly bounded in time, we use (4.73) with $2/p < s < 1, s' = 0$ so that $W_p^s(\Gamma) \hookrightarrow C(\Gamma)$ by Sobolev embedding.

$$\|\exp(-t\Lambda)u\|_{C(\Gamma)} \leq C_0 \|\exp(-t\Lambda)u\|_{W_p^s(\Gamma)} \leq C_0 M_p^{s,0}(1+t^{-s}) \|u\|_{L^p(\Gamma)}, \tag{4.77}$$

where C_0 and C_1 are positive constants. Combining inequalities (4.76) and (4.77), one obtains

$$\|\exp(-t\Lambda)\|_{\mathcal{L}(C(\Gamma))} \leq \min\left(\exp(\mu t), C_0 M_p^{s,0}(1+t^{-s})\right). \tag{4.78}$$

Hence $\|\exp(-t\Lambda)\|_{\mathcal{L}(C(\Gamma))}$ is uniformly bounded in time. □

We now state two results that hold if Ω_i consists of a single connected component. These results will only be used in Section 9, in the proof of Theorem 9.2 and Theorem 9.3. For a function u defined on Γ, define:

$$\mathcal{P}u = u - \overline{u},\ \overline{u} = \frac{1}{|\Gamma|}\int_\Gamma u(x)dS_x. \tag{4.79}$$

where $|\Gamma|$ is the area of the membrane Γ. We have the following result:

Proposition 4.15. *Suppose Ω_i is composed of a single connected component. Let $\Lambda = \Lambda_i, \Lambda_\sigma$. Let $\lambda > 0$ be the smallest non-zero eigenvalue of Λ. The semigroup $\exp(-t\Lambda)$ satisfies the following estimates:*

$$\|\exp(-t\Lambda)\mathcal{P}u\|_{L^p(\Gamma)} \leq M_p^s(1+t^{-s})\exp(-\lambda t)\|\mathcal{P}u\|_{W_p^s(\Gamma)}, \tag{4.80}$$

$$\|\exp(-t\Lambda)\mathcal{P}u\|_{C(\Gamma)} \leq M_c \exp(-\lambda t)\|\mathcal{P}u\|_{C(\Gamma)}, \tag{4.81}$$

where $0 < s < 1$, the constant M_p^s only depends on s, p and not on t, and the constant M_c does not depend on t.

We can in fact prove a generalization of (4.80) corresponding to (4.73) of the previous proposition. We do not pursue this here since we shall only make use of the special case (4.80).

Proof. Recall from the proof of the previous Proposition that the spectrum of Λ consists only of non-negative eigenvalues. The eigenspace that corresponds to the eigenvalue 0 consists of the constant functions. This can be seen from (3.20) and the fact that Ω_i and Ω_e are connected. It is clear that the operation $u \in L^2(\Gamma) \to \overline{u}$ defines an orthogonal projection $\mathcal{Q}$. Given that eigenspaces are orthogonal in $L^2(\Gamma)$, $\mathcal{Q}$ is the spectral projection corresponding to the eigenvalue 0 in $L^2(\Gamma)$. This implies that:

$$\mathcal{Q}\Lambda u = \Lambda\mathcal{Q}u. \tag{4.82}$$

for any smooth function u. Since smooth functions are dense in $L^p(\Gamma)$, the above is true for any function u in the domain of Λ seen as an operator on $L^p(\Gamma)$. For simple eigenvalue, a projection that commutes with the operator characterizes a spectral projection (p407, Lemma A.2.8 in [19]). We thus see that $\mathcal{Q}$ is, in fact, also the spectral projection corresponding to the eigenvalue 0 on $L^p(\Gamma)$. From this, it immediately follows that $\mathcal{P}$ is the spectral projection corresponding to the spectral subset $\sigma(\Lambda)\backslash\{0\}$ on $L^p(\Gamma)$ for any p.

Consider $X_p = \mathcal{P}(L^p(\Gamma))$. The operator $\exp(-t\Lambda)$ is an analytic semigroup on L^p and the smallest eigenvalue of Λ restricted X^p is λ. Since this eigenvalue is semisimple, we have the estimate (4.80).

To obtain the second inequality, we use (4.80) with $2/p < s < 1$:

$$\begin{aligned} &\|\exp(-t\Lambda)\mathcal{P}u\|_{C(\Gamma)} \le C_0 \|\exp(-t\Lambda)\mathcal{P}u\|_{W_p^s(\Gamma)} \\ \le &C_0 M_p^{s,0}(1+t^{-s})\exp(-\lambda t)\|\mathcal{P}u\|_{L^p(\Gamma)} \le C_1(1+t^{-s})\exp(-\lambda t)\|\mathcal{P}u\|_{C(\Gamma)}. \end{aligned} \tag{4.83}$$

We may combine this with the uniform bound (4.74) to obtain the desired estimate. □

We note that an estimate similar to the above can be proved even when Ω_i consists of multiple connected components by replacing $\mathcal{P}$ with the spectral projection corresponding to the eigenvalue 0.

We also make note of the following refinement of Corollary 4.6 valid when Ω_i is connected.

Corollary 4.16. *Suppose Ω_i is connected. Let $\Lambda = \Lambda_i, \Lambda_\sigma$ and u be a smooth function. Let $\|u\|_{C(\Gamma)} = M$ and suppose $u(x_0) = M$. Then,*

$$-(\Lambda u)(x_0) \le \beta(-\Lambda)\|M-u\|_{C(\Gamma)} \le 2\beta(-\Lambda)M, \tag{4.84}$$

$$-(\Lambda u)(x_0) \le 2\beta(-\Lambda)\|u-\overline{u}\|_{C(\Gamma)} = 2\beta(-\Lambda)\|\mathcal{P}u\|_{C(\Gamma)}. \tag{4.85}$$

Proof. The first inequality can be obtained by letting $\lambda_L = 0$ in (4.18). To obtain the second inequality, note that:

$$\|M-u\|_{C(\Gamma)} = \|M-\overline{u}\|_{C(\Gamma)} + \|u-\overline{u}\|_{C(\Gamma)} \le 2\|u-\overline{u}\|_{C(\Gamma)}. \tag{4.86}$$

□

5. Local Existence

We are now ready to prove a local existence result for the system (2.14). We start by introducing the notions of solution we shall use in this paper.

Let $V = (v, w) = (v, w_1, \cdots, w_N)$ and suppose $v, w_k \in B$, where B is some Banach space with values in $\mathbb{R}$. The Banach space B is either the space of k-times continuously differentiable functions, $C^k(\Gamma), k \in \mathbb{Z}^+$, the space of functions whose k-th partial derivatives are α-Hölder continuous, $C^{k+\alpha}(\Gamma),\ k \in \mathbb{Z},\ 0 < \alpha < 1$, or the Sobolev space, $W_p^s(\Gamma),\ s \ge 0,\ 1 < p < \infty$.

It is often convenient to view $w = (w_1, \cdots, w_N) \in B^N$ not as a collection of N functions with values in $\mathbb{R}$, but as a function with values in $\mathbb{R}^N$. In accordance with this view, if $w_k \in B = C(\Gamma)$, for example, we equip $w \in B^N$ with the norm:

$$\|w\|_{(C(\Gamma))^N} = \|w\|_{C(\Gamma;\mathbb{R}^N)} = \sup_{x\in\Gamma}|w(x)|, \tag{5.1}$$

where $|w(x)|$ denotes the Euclidean norm in $\mathbb{R}^N$. Note that the above is equivalent to the following norm on $w \in (C(\Gamma))^N$ which views w as a collection of N functions with values in $\mathbb{R}$:

$$\|w\|'_{(C(\Gamma))^N} = \max\left(\|w_1\|_{C(\Gamma)}, \cdots, \|w_N\|_{C(\Gamma)}\right). \tag{5.2}$$

We define the norms of $(C^k(\Gamma))^N, (C^{k+\alpha}(\Gamma))^N$ and $(W_p^s(\Gamma))^N$ similarly to (5.1). We define the norm $V = (v, w) = (v, w_1, \cdots, w_N) \in B^{N+1}$ as:

$$\|V\|_{B^{N+1}} = \max\left(\|v\|_B, \|w\|_{B^N}\right). \tag{5.3}$$

For any Banach space B, define $C([a, b]; B)$, $0 \le a < b$, to be the space of continuous functions on $[a, b]$ which take values in B. For $U \in C([a, b]; B)$, the norm on this space is:

$$\|U\|_{C([a,b];B)} = \sup_{0 \le s \le t} \|U(s)\|_B. \tag{5.4}$$

We say $U \in C^n(I; B)$, where I is an interval of the nonnegative real line, if all derivatives $U^{(k)}, 0 \le k \le n$ belong to $C([c, d]; X)$ where $[c, d]$ is an arbitrary closed interval included in I. The function U_n is said to converge to U in $C(I; B)$ if U_n converges to u in $C([c, d]; B)$ for every $[c, d] \subset I$.

Let

$$X = C(\Gamma) \text{ or } W_p^s(\Gamma), s > 2/p,\ Y = X^{N+1}. \tag{5.5}$$

The condition $s > 2/p$ implies that $W_p^s(\Gamma)$ embeds continuously into $C(\Gamma)$. Define the following evolution operator acting componentwise on $V = (v, w) = (v, w_1, \cdots, w_N) \in Y = X^{N+1}$:

$$G(t)V = (\exp(-t\Lambda)v, w) = (\exp(-t\Lambda)v, w_1, \cdots, w_N), \tag{5.6}$$

where $\Lambda = \Lambda_i$ or Λ_σ. Thus G is the identity on $w = (w_1, \cdots, w_N)$. We make the following observation.

Lemma 5.1. *When $X = W_p^s(\Gamma), s \ge 0, 1 < p < \infty$, $G(t)$ is an analytic semigroup whose generator is given by $\mathcal{L} = (-\Lambda, 0, \cdots, 0)$ where 0 denotes the zero operator acting on X. The domain of $\mathcal{L}$ is given by $W_p^{s+1}(\Gamma) \times (W_p^s(\Gamma))^N$.*

Proof. This is an immediate consequence of Proposition 4.13. □

For the smooth function $F = (f, g_1, \cdots, g_N)$ taking $\mathbb{R}^{N+1}$ to $\mathbb{R}^{N+1}$, define the Nemytskij operators $V \mapsto F^{\mathrm{N}}(V), V \mapsto g^{\mathrm{N}}(V)$ as follows:

$$\begin{aligned} (F^{\mathrm{N}}(V))(x) &= ((f^{\mathrm{N}}(V))(x), (g^{\mathrm{N}}(V))(x)) = (f(V(x)), g(V(x))), \\ g^{\mathrm{N}}(V) &= (g_1^{\mathrm{N}}(V), \cdots, g_N^{\mathrm{N}}(V)). \end{aligned} \tag{5.7}$$

We shall see shortly that $f^{\mathrm{N}}, g_k^{\mathrm{N}}$ are well-defined as maps from Y to X and hence F^{N} as a map from Y to Y. In the proof of Lemma 5.7, it is useful to make a notational distinction between the smooth functions F, f, g, g_k and the corresponding Nemytskij operators $F^{\mathrm{N}}, f^{\mathrm{N}}, g^{\mathrm{N}}, g_k^{\mathrm{N}}$. In all other places in this paper, we shall not make this notational distinction.

We can now define the solution classes we will be concerned with in this paper. These definitions are in accordance with the usual solution concepts for semilinear parabolic equations [19, 27].

Definition 5.2 (Mild solution)**.** *A function $U \in C([0, T]; Y), T > 0$, where Y is defined in* (5.5)*, is a* mild solution *of system* (2.14) *with initial condition* (2.2) *if U satisfies the following integral equation:*

$$\begin{aligned} U(t) &= G(t)U_0 + \int_0^t G(t-s)F^N(U(s))ds, \\ U_0 &= (v^0, w^0) = (v^0, w_1^0, \cdots, w_N^0) \in Y, \end{aligned} \tag{5.8}$$

for all $t \in [0, T]$.

Definition 5.3 (Classical and strict solutions)**.** *Let* $X = W_p^s(\Gamma), s > 2/p$ *in* (5.5). $U = (v, w) = (v, w_1, \cdots, w_k) \in C([0, T]; Y)$ *is a* classical solution *of system* (2.14) *with initial condition* (2.2) *if* $U \in C^1((0, T]; Y)$*,* $v \in C((0, T]; W_p^{s+1}(\Gamma))$ *and*

$$\frac{\partial v}{\partial t} = -\Lambda v + f^N(U(t)) \quad \text{for } 0 < t \leq T, \tag{5.9a}$$

$$\frac{\partial w}{\partial t} = g^N(U(t)) \quad \text{for } 0 < t \leq T, \tag{5.9b}$$

$$U(0) = U_0 \in Y. \tag{5.9c}$$

A classical solution U *is called a* strict solution *if* $U(t) \in C^1([0, T]; Y)$*,* $v \in C([0, T]; W_p^{s+1}(\Gamma))$ *and the equations* (5.9a) *and* (5.9b) *are satisfied up to* $t = 0$.

We prove some estimates regarding F^{N}.

Lemma 5.4. *Let* $X = C(\Gamma)$ *or* $W_p^s(\Gamma)$ *where* $s > 2/p$*. Suppose* $V \in Y = X^{N+1}$*. Take any* $R > 0$*, and suppose* $V \in Y, \|V\|_{(C(\Gamma))^{N+1}} \leq R$*. We have*

$$\left\|F^{\mathrm{N}}(V)\right\|_Y \leq C_0 + C_1(R) \|V\|_Y, \tag{5.10}$$

where C_0 *is a constant that depends only on* F *and* C_1 *is a constant that depends only on* R *and the derivatives of* F*. Suppose* $V, V' \in X^{N+1}$ *and* $\|V\|_Y, \|V'\|_Y \leq R$*. Then,*

$$\left\|F^{\mathrm{N}}(V) - F^{\mathrm{N}}(V')\right\|_Y \leq C_2(R) \|V - V'\|_Y, \tag{5.11}$$

where C_2 *depends only on* R *and the derivatives of* F*.*

Proof. When $X = C(\Gamma)$ the proof is immediate. When $X = W_p^s(\Gamma), s > 2/p$, we use the following estimate (Theorem 1 of p. 387 in [32]). Let the smooth functions F take $\mathbf{0} \in \mathbb{R}^{N+1}$ to itself. For $V \in W_p^s(\mathbb{R}^2)^{N+1}, \|V\|_{L^\infty(\mathbb{R}^2)^{N+1}} \leq R$,

$$\left\|F^{\mathrm{N}}(V)\right\|_{W_p^s(\mathbb{R}^2)^{N+1}} \leq \tilde{C}_1 \|V\|_{W_p^s(\mathbb{R}^2)^{N+1}}, \tag{5.12}$$

where $\tilde{C}_1$ depends only on R and the derivatives of F. Using a partition of unity on the compact surface Γ, we see immediately that this yields the desired bound (5.10) when $F(\mathbf{0}) = \mathbf{0}$. In the general case, we can apply this bound to the Nemytskij operator defined by the function $F - F(\mathbf{0})$.

We now turn to (5.11). We prove this bound component by component. First, note the Hadamard identity:

$$f(x) - f(y) = \int_0^1 \nabla f(sx + (1-s)y) \cdot (x - y) ds, \tag{5.13}$$

where $x, y \in \mathbb{R}^{N+1}$. We find:

$$f^{\mathrm{N}}(V) - f^{\mathrm{N}}(V') = \int_0^1 (\nabla f)^{\mathrm{N}}(sV + (1-s)V') \cdot (V - V') ds, \tag{5.14}$$

where f^{N} and $(\nabla f)^{\mathrm{N}}$ are the Nemytskij operator corresponding to f and ∇f respectively. Taking the $X = W_p^s$ norm on both sides and noting that $W_p^s, s > 2/p$, satisfies the Banach algebra property, we see that:

$$\left\|f^{\mathrm{N}}(V) - f^{\mathrm{N}}(V)\right\|_X \leq C_3 \int_0^1 \left\|(\nabla f)^{\mathrm{N}}(sV + (1-s)V')\right\|_Y \|V - V'\|_Y \, ds, \tag{5.15}$$

where C_3 is a constant that depends only on Γ. Using (5.10),

$$\left\|(\nabla f)^{\mathrm{N}}(sV+(1-s)V')\right\|_Y \leq C_4\left\|sV+(1-s)V'\right\|_Y, \tag{5.16}$$

where C_4 depends only on $\|sV+(1-s)V'\|_{(C(\Gamma))^{N+1}}$ and the derivatives of f. Note

$$\begin{aligned}\|sV+(1-s)V'\|_{(C(\Gamma))^{N+1}} &\leq s\|V\|_{(C(\Gamma))^{N+1}}+(1-s)\|V'\|_{(C(\Gamma))^{N+1}}\\ &\leq \max(\|V\|_{(C(\Gamma))^{N+1}},\|V'\|_{(C(\Gamma))^{N+1}}),\end{aligned} \tag{5.17}$$

and likewise for $\|sV+(1-s)V'\|_Y$. Since $Y \hookrightarrow (C(\Gamma))^{N+1}$ we have

$$\left\|(\nabla f)^{\mathrm{N}}(sV+(1-s)V')\right\|_Y \leq C_5, \tag{5.18}$$

where C_5 depends only on $\max(\|V\|_Y, \|V'\|_Y)$ and the derivatives of f. Combining this with (5.15), we have:

$$\left\|f^{\mathrm{N}}(V)-f^{\mathrm{N}}(V)\right\|_X \leq C_3C_5\|V-V'\|_Y. \tag{5.19}$$

We can use the same argument to obtain analogous bounds for $g_k, k=1,\cdots N$ in place of f. □

Lemma 5.1 and Lemma 5.4 allow us to apply standard results on abstract semilinear parabolic equations to the system (2.14).

Theorem 5.5 (Local existence)**.** *Let $X = C(\Gamma)$ or $W_p^s(\Gamma)$. For any initial data $U_0 \in Y$, there exists a unique mild solution $U \in C([0,T];Y)$ for some positive T that depends only on $\|U_0\|_Y$. Moreover, the solution depends continuously on initial data:*

$$\|U-U'\|_{C([0,T];Y)} \leq C\|U_0-U_0'\|_Y, \tag{5.20}$$

where U, U' are the solutions to the initial value problem with initial data U_0 and U_0' respectively and C depends on the norm of $\max\{\|U_0\|, \|U_0'\|_Y\}$. *When $X = W_p^s(\Gamma)$, the notion of mild and classical solutions are equivalent. In addition, if $U_0 = (v^0, w^0)$ and $v^0 \in W_p^{s+1}$, U is a strict solution.*

Proof. The existence, uniqueness and continuous dependence on initial data follow from a standard fixed point argument. The statement on mild, classical and strict solutions follows from the fact that $G(t)$ is an analytic semigroup. See, for example, [19, 27] for details. □

Let us now comment on the single equation case (2.15), where $N = 0$ and the gating variables w_k are absent. We may define the mild, classical and strict solutions analogously to (5.8) and (5.9a) as follows. For $X = W_p^s(\Gamma), s > 2/p$ or $C(\Gamma)$, a function $v \in C([0,T];X)$ is a called mild solution if it satisfies:

$$v(t) = \exp(-t\Lambda)v^0 + \int_0^t \exp(-\Lambda(t-s))f(v(s))ds \tag{5.21}$$

for all $t \in [0,T]$ and $v^0 \in X$. We call v a classical solution if $v \in C([0,T],X) \cap C^1((0,T];W_p^s(\Gamma)) \cap C((0,T],W_p^{s+1}(\Gamma))$ satisfying the equation:

$$\frac{\partial v}{\partial t} = -\Lambda v + f(v), \text{ for } 0 < t \leq T \tag{5.22}$$

and $v(0) = v^0 \in X$. Strict solutions may be defined in a similar fashion. The main difference between the following proposition and Theorem 5.5 is that v is smooth for $v > 0$ due to the regularizing effect of the semigroup $\exp(-t\Lambda)$.

Proposition 5.6. *For any initial data $v^0 \in X$, equation* (2.15) *a unique mild solution $v \in C([0,T],X)$ for some positive T. When $X = W_p^s(\Gamma), s > 2/p$, the solution is classical. If $v^0 \in W_p^{s+1}(\Gamma)$, the solution is strict. The solution depends continuously on initial data, analogously to* (5.20)*. Moreover, the solution is in $C^\infty(\Gamma)$ for $t > 0$.*

Proof. When $X = C(\Gamma)$, the claim is the same as Theorem 5.5. When $X = W_p^s$, we first note by Proposition 4.13 that $-\Lambda$ generates an analytic semigroup on X with domain W_p^{s+1}. We may apply the theory of analytic semigroups [19, 27] to reach the desired conclusion. The smoothness of the solution follows since f is a smooth function. □

We end this section by noting the following comparison principle.

Proposition 5.7 (Comparison principle)**.** *Suppose $-\Lambda$ satisfies the positivity principle. Let v and $\tilde{v}$ be mild solutions to* (2.15) *where the initial conditions v^0 and $\tilde{v}^0$ are continuous functions on Γ. If $\tilde{v}^0 \geq v^0$, then $\tilde{v}(t) \geq v(t)$ for any positive t for which $\tilde{v}(t)$ and $v(t)$ both exist. The same conclusion holds if we replace the above with the strict inequality.*

Proof. The proof is standard and elementary, but we give its outline here so that the reader can compare it with an argument for the quasipositivity case which we will discuss in Section 6.2.

Note first that since v^0 and $\tilde{v}^0$ are both in $C(\Gamma)$, they are also in $L^p(\Gamma), p > 2$. Therefore, by the previous proposition, the solution u and v are smooth for $t > 0$ and satisfy the equation (2.15) pointwise.

Suppose both solutions v and $\tilde{v}$ exist up to time $t = T$. Then the difference $u = \tilde{v} - v$. u solves the equation:

$$\frac{\partial u}{\partial t} = -\Lambda u + g(t)u,\ g(t) = \frac{f(\tilde{v}) - f(v)}{\tilde{v} - v}. \tag{5.23}$$

where $0 < t \leq T$. Note that $g(t)$ is a smooth function in x and is continuous in t. Take any constant M such that

$$M - \sup_{0<t\leq T} \|g(t)\|_{C(\Gamma)} \equiv K > 0. \tag{5.24}$$

Consider the function $\tilde{u}(t) = \exp(Mt)u(t)$. Now suppose that $\tilde{u}(0) = \tilde{v}^0 - v^0 > 0$. Since Γ is compact, $\tilde{u}(0) > \epsilon$ for some $\epsilon > 0$. We now show that $\tilde{u}(t) > \epsilon$ for $0 < t \leq T$. For assume otherwise. Define

$$t_\epsilon = \sup_{t\in\mathcal{A}} t,\ \mathcal{A} = \{t \in (0,T] | \tilde{u}(s) > \epsilon \text{ if } s < t\}. \tag{5.25}$$

Note that $t_\epsilon > 0$ since $\tilde{u}(t)$ is continuous in t and x and $\tilde{u}(0) > \epsilon$. At $t = t_\epsilon$, there is a point $x_\epsilon \in \Gamma$ at which $\tilde{u}(t_\epsilon, x_\epsilon) = \epsilon$. Then the derivative of $\frac{\partial u}{\partial t}$ at x_ϵ:

$$\begin{aligned} \left.\frac{\partial \tilde{u}}{\partial t}\right|_{x=x_\epsilon, t=t_\epsilon} &= -(\Lambda u)(x_\epsilon, t_\epsilon) + (M+g)u(x_\epsilon, t_\epsilon) \\ &\geq -(\Lambda u)(x_\epsilon, t_\epsilon) + K\epsilon = -(\Lambda(u - \epsilon 1_\Gamma))(x_\epsilon, t_\epsilon) + K\epsilon \geq K\epsilon > 0, \end{aligned} \tag{5.26}$$

where 1_Γ is the function that is equal to 1 everywhere on Γ. In the first inequality, we used (5.24), in the second equality, we used $\Lambda 1_\Gamma = 0$. In the last inequality, we used the positivity of $-\Lambda$. We see therefore, that $\frac{\partial \tilde{u}}{\partial t} > 0$, but this is a contradiction because this implies the existence of $t_0 < t_\epsilon$ such that $u(t_0) \ngeq \epsilon$. We thus see that $\tilde{u}(0) = u(0) > 0$ implies $\tilde{u}(t) = \exp(Mt)u(t) > 0$. Thus, $u(0) > 0$ implies $u(t) > 0$. When $\tilde{v}(0) - \tilde{v}(0) \geq 0$, we may consider the solutions $\tilde{v}_\epsilon(t)$ corresponding to the initial conditions $\tilde{v}_\epsilon(0) = \tilde{v}^0 + \epsilon, \epsilon > 0$. From what we proved above, $\tilde{v}_\epsilon - v > 0$. By the previous proposition, the solution depends continuously on initial data in $C(\Gamma)$, and thus, we may take the limit $\epsilon \to 0$ to conclude that $\tilde{v} - v \geq 0$. □

6. Global Existence for FitzHugh-Nagumo(FN) type Systems

In this section, we establish global existence of solutions to system (2.14) for a general class of nonlinearities $F = (f, g)$ which includes the FitzHugh-Nagumo nonlinearity (see (6.27a) and (6.27b)). We also discuss asymptotic uniform bounds, the meaning of which will be clarified later. We begin with the following proposition, which holds regardless of the type of nonlinearity F:

Proposition 6.1. *Let $X = C(\Gamma)$ or $W_p^s(\Gamma), s > 2/p$, and let $U(t)$ be a mild solution in $C([0, T); Y)$ where $T < \infty$ is the maximal interval of existence. Then,*

$$\lim_{t \to T} \|U(t)\|_{(C(\Gamma))^{N+1}} = \infty. \tag{6.1}$$

Proof. By Lemma 5.4, F^{N} maps bounded sets of Y into bounded sets of Y. Therefore [19],

$$\lim_{t \to T} \|U(t)\|_Y = \infty. \tag{6.2}$$

When $X = C(\Gamma)$, the claim is thus trivial. When $X = W_p^s(\Gamma)$, we argue by contradiction. Suppose the claim is not true and $\|U(t)\|_{(C(\Gamma))^{N+1}}$ remains bounded on $[0, T)$. Take the Y norm on both sides of (5.8).

$$\begin{aligned} \|U(t)\|_Y &\leq M \|U_0\|_Y + \int_0^t M(C_0 + C_1 \|U(s)\|_Y) ds, \\ M &= \sup_{0<t<T} \|G(t)\|_{\mathcal{L}(Y)}, \ 0 < t < T, \end{aligned} \tag{6.3}$$

where we used the boundedness of the semigroup $G(t)$, (5.10) of Lemma 5.4 and the assumption that $\|U(t)\|_{(C(\Gamma))^{N+1}}$ remains bounded. By Gronwall's inequality, we see immediately that $\|U(t)\|_Y$ must also remain bounded in $0 < t < T$. This is a contradiction. □

We see that a priori bounds in the maximum norm imply global existence.

6.1. Global Existence and the κ-Contracting Rectangle.

In this subsection we discuss global existence. The key to our discussion is the following.

Definition 6.2 (κ-Contracting rectangle)**.** *Consider the following rectangular region in $\mathcal{R} \subset \mathbb{R}^{N+1}$:*

$$\mathcal{R} = \{y \in \mathbb{R}^{N+1} |\, L_k^- \leq y_k \leq L_k^+, \ 0 \leq k \leq N, \ y = (y_0, y_1, \cdots, y_N)\}, \tag{6.4}$$

*where we shall take $-L_0^- = L_0^+ = L_0 > 0$. We call $\mathcal{R}$ a κ-*contracting rectangle *if it satisfies the following properties.*

$$\begin{aligned} &f(y) > \kappa y_0, \ f(y) < -\kappa y_0 \ \textit{resp. at } y_0 = -L_0, \ y_0 = L_0, \\ &\quad g_k(y) > 0, \ g_k(y) < 0 \ \textit{resp. at } y_k = L_k^-, \ y_k = L_k^+, \ k = 1, \cdots, N. \end{aligned} \tag{6.5}$$

When $\kappa = 0$, this reduces to the definition of the contracting rectangle found in [30, 34]. Given a rectangle $\mathcal{R}$, we define the following subset of $(C(\Gamma))^{N+1}$:

$$\mathcal{U}_{\mathcal{R}} = \{V \in (C(\Gamma))^{N+1} | \, V(x) \in \mathcal{R} \text{ for all } x \in \Gamma\}. \tag{6.6}$$

Lemma 6.3. *Let $\mathcal{R}$ be a κ-contracting rectangle with $\kappa = 2\beta(-\Lambda)$, and let $U(t)$ be a mild solution of (2.14) in $C([0,T];Y)$ such that the initial data U_0 is in $\mathcal{U}_{\mathcal{R}}$. Then $U(t) \in \mathcal{U}_{\mathcal{R}}$ for all $t \in [0,T)$.*

Proof. We first let the initial data U_0 be a smooth function whose values $U_0(x), x \in \Gamma$ lie in the interior of the rectangle $\mathcal{R}$. It is clear from Theorem 5.5 that $U(t)$ is a smooth function for $t \in [0,T)$ and that $U(t)$ is a classical solution satisfying equation (2.14a) and (2.14b) pointwise.

We argue by contradiction. Suppose $U(t)$ leaves $\mathcal{U}_{\mathcal{R}}$ before time T. Let

$$t_{\mathcal{R}} = \inf_{t \in \mathcal{A}} t, \; \mathcal{A} = \{t \in [0,t_0) | \, U(t) \notin \mathcal{U}_{\mathcal{R}}\}. \tag{6.7}$$

Since U is continuous in (t,x) and Γ is compact, there are points $z \in \Gamma$ for which $U(t_{\mathcal{R}}, z) = (v(t_{\mathcal{R}}, z), w_1(t_{\mathcal{R}}, z), \cdots, w_n(t_{\mathcal{R}}, z))$ lies on the boundary of $\mathcal{R}$. Since $U(0,x) = U_0(x), x \in \Gamma$ lies in the interior of $\mathcal{R}$, $t_{\mathcal{R}} > 0$. Suppose $w_k(t_{\mathcal{R}}, z) = L_k^+$. Then,

$$\left.\frac{\partial w_k}{\partial t}\right|_{(t_{\mathcal{R}}, z)} = g_k(U(t_{\mathcal{R}}, z)) < 0. \tag{6.8}$$

But this is clearly a contradiction since $w_k(t,z) \leq L_k^+$ for $t < t_{\mathcal{R}}$. Likewise, $w_k(t_{\mathcal{R}}, z) = L_k^-$ also leads to a contradiction.

Now suppose $v(t,z) = L_0$.

$$\left.\frac{\partial v}{\partial t}\right|_{(t_{\mathcal{R}}, z)} = -(\Lambda v)(t_{\mathcal{R}}, z) + f(U(t_{\mathcal{R}}, z)) \leq 2\beta(-\Lambda)L_0 - \kappa L_0 < 0, \tag{6.9}$$

where we used Corollary 4.16. This is a contradiction

In the general case when U_0 is not smooth and its values may not lie in the interior of $\mathcal{R}$, we use an approximation argument. Approximate U_0 by a sequence $V_{0,k}$ of smooth functions so that $V_{0,k} \to U_0$ in Y. We can even make it so that $V_{0,k}(x), x \in \Gamma$ lies in the interior of $\mathcal{R}$. Let V_k be the mild solution whose initial values is $V_{0,k}$. By what we proved above, V_k lies in $\mathcal{U}_{\mathcal{R}}$ for $0 \leq t < T$. By Theorem 5.5, we have:

$$\|V_k - V_l\|_{C([0,T];Y)} \leq C \|V_{0,k} - V_{0,l}\|_Y \tag{6.10}$$

for some constant C. Thus, V_k forms a Cauchy sequence in $C([0,T];Y)$. Since $C([0,T];Y)$ is complete, V_k has a limit $U \in C([0,T];Y)$. Substitute $V_{0,k}$ and V_k in (5.8):

$$V_k(T) = G(T)V_{0,k} + \int_0^T G(T-s)F(V_k(s))ds. \tag{6.11}$$

Using $V_k \to U$, $V_{0,k} \to U_0$ and (5.11), we conclude:

$$U(T) = G(T)U_0 + \int_0^T G(T-s)F(U(s))ds. \tag{6.12}$$

This shows that we can solve the initial value problem with $U_0 \in Y$ up to any $T > 0$. Since $\mathcal{U}_{\mathcal{R}} \cap Y$ is a closed set, and $V_k \in \mathcal{U}_{\mathcal{R}} \cap Y$, $U(t) \in \mathcal{U}_{\mathcal{R}}$. □

We now state our first global existence result.

Theorem 6.4 (Global existence)**.** *Let $\mathcal{R}$ be a κ-contracting rectangle with $\kappa = 2\beta(-\Lambda)$. Let the initial data U_0 be in $\mathcal{U}_{\mathcal{R}} \cap Y$. Then* (2.14) *has a unique mild solution $U \in C([0,\infty);Y)$ and $U(t) \in \mathcal{U}_{\mathcal{R}}$ for all $t \geq 0$.*

Proof. The statement follows from Proposition 6.1 and the previous lemma. □

6.2. **Asymptotic Uniform Bounds.** We now discuss asymptotic uniform bounds.

Definition 6.5 (Time-dependent κ-contracting rectangle)**.** *Let I denote an interval of the form $[0,T], T > 0$ or of the form $[0,\infty)$. We say that*

$$\mathcal{R} = \{y \in \mathbb{R}^{N+1} |\, L_k^-(t) \leq y_k \leq L_k^+(t),\ 0 \leq k \leq N,\ y = (y_0, y_1, \cdots, y_N)\} \quad (6.13)$$

is a time-dependent κ-contracting rectangle *if the following conditions hold for $t \in I$:*

$$f(y) > \kappa y_0 - L_0'(t) \text{ at } y_0 = -L_0(t), \quad (6.14)$$

$$f(y) < -\kappa y_0 + L_0'(t) \text{ at } y_0 = L_0(t), \quad (6.15)$$

$$g_k(y) > (L_k^-)'(t) \text{ at } y_k = L_k^-(t),\ k = 1, \cdots, N, \quad (6.16)$$

$$g_k(y) < (L_k^+)'(t) \text{ at } y_k = L_k^+(t),\ k = 1, \cdots, N. \quad (6.17)$$

where $L_0'(t)$ and $(L_k^{\pm})'(t)$ denote time derivatives.

Given a time-dependent κ-contracting rectangle $\mathcal{R}(t)$, we define the following subset of $(C(\Gamma))^{N+1}$, similarly to (6.6):

$$\mathcal{U}_{\mathcal{R}}(t) = \{V \in (C(\Gamma))^{N+1} |\, V(x) \in \mathcal{R}(t) \text{ for all } x \in \Gamma\}. \quad (6.18)$$

Lemma 6.6. *Let I be an interval of the form $[0,T]$ or of the form $[0,\infty)$, and let $\mathcal{R}(t), t \in I$ be a time-dependent κ-contracting rectangle as defined above, with $\kappa = 2\beta(-\Lambda)$. Let $U(t)$ be a mild solution of* (2.14) *in $(C(\Gamma))^{N+1}$ whose initial data U_0 lies in $\mathcal{U}_{\mathcal{R}}(0)$. Then $U(t) \in \mathcal{U}_{\mathcal{R}}(t)$ for all $t \in I$.*

Proof. The proof is essentially the same as that of Lemma 6.3. Suppose the initial data U_0 is smooth and $U_0(x), x \in \Gamma$ lie in the interior of $\mathcal{R}(0)$. The mild solution $U(t)$ is smooth and therefore we may use pointwise arguments.

We argue by contradiction. Define $t_{\mathcal{R}} > 0$ as in (6.7). Let $z \in \Gamma$ be a point at which $U(t,z)$ hits the boundary of $\mathcal{R}(t_{\mathcal{R}})$. Suppose $w_k(t_{\mathcal{R}}, z) = L_k^+(t_{\mathcal{R}})$. By the definition of the time-dependent κ-contracting rectangle, we see that:

$$\left.\frac{\partial}{\partial t}(w_k - L_k^+)\right|_{(t_{\mathcal{R}},z)} = g_k(U(t_{\mathcal{R}}, z)) - (L_k^+)'(t_{\mathcal{R}}) < 0. \quad (6.19)$$

This is contradiction since $w_k - L_k^+ < 0$ for $t < t_{\mathcal{R}}$. An analogous argument can be made if $w_k(t_{\mathcal{R}}, z) = L_k^-(t_{\mathcal{R}})$. If $v(t_{\mathcal{R}}, z) = L_0(t_{\mathcal{R}})$, we have:

$$\left.\frac{\partial}{\partial t}(v - L_0)\right|_{(t_{\mathcal{R}},z)} \leq 2\beta(-\Lambda)L_0(t_{\mathcal{R}}) + f(U(t_{\mathcal{R}}, z)) - L_0'(t_{\mathcal{R}}) < 0. \quad (6.20)$$

We used $\kappa = 2\beta(-\Lambda)$ and the definition of the time-dependent κ-contracting rectangle. This is again a contradiction since $v - L_0 < 0$ for $t < t_{\mathcal{R}}$. We may argue analogously if $v(t_{\mathcal{R}}, z) = -L_0(t_{\mathcal{R}})$.

For general initial data, we may argue by approximation. □

Theorem 6.7 (Asymptotic uniform bounds)**.** *Let $B_k(M), k = 1, \ldots, N$ be C^1 functions defined on $[M_*, \infty)$ such that $B_k'(M) > 0$ and $B_k(M) \to \infty$ as $M \to \infty$. Suppose that the rectangle defined by setting $L_k^{\pm} = \pm B_k(M)$ and $L_0 = M$ in* (6.2) *is a κ-contracting rectangle with $\kappa = 2\beta(-\Lambda)$ for each $M \in [M_*, \infty)$. Then any mild solution $U(t) = (v(t), w_1(t), \ldots, w_N(t))$ of* (2.14) *exists globally for $t \geq 0$. Moreover, there is a time $T < \infty$ that depends only on the L^∞ norm of the initial data such that*

$$\|v(t)\|_{L^\infty(\Gamma)} \leq M_*, \ \|w_k(t)\|_{L^\infty(\Gamma)} \leq B_k(M_*) \tag{6.21}$$

for $t \geq T$.

Proof. Let the initial value $U_0(x), x \in \Gamma$ be contained in a κ-contracting rectangle defined by $M = M_0 \geq M_*$. Such an M_0 exists since $B_k(M) \to \infty$ as $M \to \infty$. If $M_0 = M_*$, the statement of the theorem is trivially true by Theorem 6.4. We thus assume $M_0 > M_*$. Let $\mathcal{R}_M$ be the κ-contracting rectangle that corresponds to M. Pick a $\delta > 0$ such that the following hold for all $M \in [M_*, M_0]$ and $y = (y_0, y_1, \ldots, y_N) \in \partial\mathcal{R}_M$:

$$f(y) + \kappa y_0 > \delta \text{ for } y_0 = -M, \tag{6.22}$$

$$f(y) + \kappa y_0 < -\delta \text{ for } y_0 = M, \tag{6.23}$$

$$g_k(y) > B_k'(M)\delta \text{ for } y_k = -B_k(M), \tag{6.24}$$

$$g_k(y) < -B_k'(M)\delta \text{ for } y_k = B_k(M). \tag{6.25}$$

Such a δ can be taken independent of M since the left hand side of the above inequalities are continuous functions and $B_k'(M)$ is bounded on $[M_*, M_0]$. Let

$$L_0(t) = M_U - \delta t, \ L_k(t) = B_k(L_0(t)), \ L_k^{\pm}(t) = \pm L_k(t), \tag{6.26}$$

where $t \in [0, T], T = (M_0 - M_*)/\delta$. It is easily seen that the above functions define a time-dependent κ-contracting rectangle. By Lemma 6.6, $U(T, x), x \in \Gamma$ lies in the κ-contracting rectangle $\mathcal{R}_{M_*}$. Since $U(t), t > T$ can be seen as the mild solution of (2.14a) and (2.14b) with $U(T)$ as the initial data, we may apply Lemma 6.3 to conclude that $U(t, x), x \in \Gamma$ lies in $\mathcal{R}_{M_*}$ for $t \geq T$. Note that T depends only on M_0, and hence only on the L^∞ norm of U_0. □

Consider the cubic FitzHugh-Nagumo system:

$$\frac{\partial v}{\partial t} = -\Lambda v + f_{\text{FN}}(v) - w, \tag{6.27a}$$

$$f_{\text{FN}}(v) = -Av(v - \alpha)(v - 1), \ A > 0, \ 0 < \alpha < 1,$$

$$\frac{\partial w}{\partial t} = \theta v - \mu w, \ \theta > 0, \ \mu > 0. \tag{6.27b}$$

Corollary 6.8. *The FitzHugh-Nagumo system has a global bounded solution for any initial data $U_0 \in X^2$. For $X = W_p^s(\Gamma), s > 2/p$, the solution is classical. Moreover, there are positive constants M_v and M_w such that*

$$\|v(t)\|_{C(\Gamma)} \leq M_v, \ \|w(t)\|_{C(\Gamma)} \leq M_w, \tag{6.28}$$

for $t \geq T$. This T depends only on the $C(\Gamma)$ norm of the initial data.

Proof. Let $\kappa = 2\beta(-\Lambda)$ and take any constant $q > \theta/\mu$. Take $M_* > 1$ large enough so that $f_{\text{FN}}(M_*) + (\kappa + q)M_* < 0$. This is possible thanks to the cubic growth of $f_{\rm FN}$. It is then easy to check that $L_0 = M, L_1^{\pm} = \pm B_1(M) = \pm qM$ is a κ-contracting rectangle for each $M \in [M_*, \infty)$. We may thus apply Theorem 6.7 to reach the desired conclusion. □

In view of the above result, we define the following class of nonlinearities:

Definition 6.9 (FitzHugh-Nagumo(FN) type system)**.** *Suppose the nonlinearity $F = (f, g)$ in* (2.14) *satisfies the hypotheses of Proposition 6.7. Then, we say that system* (2.14) *or the nonlinearity F is of* FitzHugh-Nagumo(FN) type*.*

7. Global Existence for Hodgkin-Huxley(HH) type Systems

Most, if not all, electrophysiology models (i.e., the nonlinear functions f and g in (2.14)) admit a contracting rectangle. Thus, if $-\Lambda$ satisfies the positivity principle, we have global uniformly bounded solutions. In the general case, we saw that if $F = (f, g)$ is of FN type, (2.14) has global uniformly bounded solutions. To show that the FitzHugh-Nagumo system falls into this class we used the fact that f grows supralinearly in magnitude for large values of v (see proof of Corollary 6.8). The FN class excludes the important case of the Hodgkin-Huxley model in which f grows only linearly with v. We shall now use energy estimates to prove a theorem that addresses this case.

We start with a definition.

Definition 7.1 (Invariant cylinder)**.** *Consider the following subset $\mathcal{C} \subset \mathbb{R}^{N+1}$:*

$$\mathcal{C} = \{y \in \mathbb{R}^{N+1} |\, L_k^- \le y_k \le L_k^+,\ 1 \le k \le N,\ y = (y_0, y_1, \cdots, y_N)\}. \tag{7.1}$$

We call $\mathcal{C}$ an invariant cylinder *if the following conditions are satisfied.*

$$g_k(y) > 0,\ g_k(y) < 0 \ \textit{resp. at}\ y_k = L_k^-,\ y_k = L_k^+,\ k = 1, \cdots, N. \tag{7.2}$$

The set $\mathcal{C}$ is thus an invariant cylinder if the vector field $(f(y), g_1(y), \cdots, g_N(y))$ be points inward on the boundary of $\mathcal{C}$. We shall refer to (7.2) as the invariant cylinder condition. For a given $M > 0$, we define, for future use, the truncated invariant cylinder $\mathcal{C}_M$ and its subsets $\mathcal{B}_M^{\pm}$:

$$\mathcal{C}_M = \mathcal{C} \cap \{y \in \mathbb{R}^{N+1} |\, |y_0| \le M)\},\ \mathcal{B}_M^{\pm} = \mathcal{C} \cap \{y \in \mathbb{R}^{N+1} | y_0 = \pm M\}. \tag{7.3}$$

Now we define the following class of nonlinearities.

Definition 7.2 (Hodgkin-Huxley(HH) type system)**.** *Suppose $F = (f, g)$ in system* (2.14) *possesses an invariant cylinder $\mathcal{C}$. We say that system* (2.14) *or the nonlinearity $F = (f, g)$ is of Hodgkin-Huxley(HH) type if the following conditions are satisfied:*

$$f \text{ is globally Lipschitz in } \mathcal{C}, \tag{7.4}$$

$$f \text{ has the following decomposition in } \mathcal{C} :$$

$$f(y) = f_1(y) + f_2(y),\ f_1(y) y_0 \le -\gamma y_0^2,\ |f_2(y)| \le \eta,\ y \in \mathcal{C}, \tag{7.5}$$

where γ and η are positive constants.

Let

$$\mathcal{U}_{\mathcal{C}} = \{V \in (C(\Gamma))^{N+1} |\, V(x) \in \mathcal{C} \text{ for all } x \in \Gamma\}, \tag{7.6}$$

$$\mathcal{U}_M = \{V \in (C(\Gamma))^{N+1} |\, V(x) \in \mathcal{C}_M \text{ for all } x \in \Gamma\}. \tag{7.7}$$

For HH type systems, we shall only consider initial values in $\mathcal{U}_{\mathcal{C}}$. We have the following result for HH type systems.

Theorem 7.3 (Global existence and asymptotic uniform bounds)**.** *Suppose system (2.14) is of HH type. Let $U_0 \in \mathcal{U}_{M_0} \cap Y$ for some $M_0 > 0$. Then, a mild solution $U(t)$ of (2.14) with initial value U_0 exists for all time, and $U(t) \in \mathcal{U}_M, 0 \leq t < \infty$ where M depends only on M_0.*

Moreover, there is a time $T > 0$ that depends only on M_0 such that $U(t) \in \mathcal{U}_{M_*}$ where $M_* > 0$ is a universal constant.

We first state a simple lemma.

Lemma 7.4. *Let H be a Hilbert space and let $a(t) \in C([0,T], H) \cap C^1((0,T]; H), T > 0$. Suppose $a(t)$ satisfies:*

$$\frac{1}{2}\frac{d}{dt}\|a\|_H^2 \leq \beta \|a\|_H - \alpha \|a\|_H^2, \ \alpha, \beta > 0, \ 0 < t \leq T, \tag{7.8}$$

where $\|\cdot\|_H$ denotes the norm in H. Then,

$$\|a(t)\|_H \leq \|a(0)\|_H \exp(-\alpha t) + \frac{\beta}{\alpha}(1 - \exp(-\alpha t)). \tag{7.9}$$

for $0 \leq t \leq T$.

Proof. If $\|a(t)\|_H = 0$, we have nothing to prove, so we assume $\|a(t)\|_H > 0$. Suppose $\|a(s)\|_H \neq 0$ for $0 < s < t \leq T$. Then, (7.8) implies:

$$\frac{d}{ds}\|a(s)\|_H \leq \beta - \alpha \|a(s)\|_H, \ 0 < s \leq t. \tag{7.10}$$

Solving this differential inequality for $0 < \epsilon \leq s \leq t$ yields:

$$\|a(t)\|_H \leq \|a(\epsilon)\|_H \exp(-\alpha(t-\epsilon)) + \frac{\beta}{\alpha}(1 - \exp(-\alpha(t-\epsilon))). \tag{7.11}$$

Noting that $a(t)$ is continuous at $t = 0$, we may take the limit $\epsilon \to 0$ in the above to obtain (7.9). Suppose $\|a(s)\|_H = 0$ for some values of $0 < s < t \leq T$. Let:

$$s_0 = \max_{s \in \mathcal{Z}} s, \ \mathcal{Z} = \{s | 0 < s < t, \|a(s)\|_H = 0\}. \tag{7.12}$$

Then, (7.10) is valid for $s_0 < s \leq t$. Solving (7.10) between $s_0 + \epsilon \leq s \leq t$, we find:

$$\|a(t)\|_H \leq \|a(s_0+\epsilon)\|_H \exp(-\alpha(t-s_0-\epsilon)) + \frac{\beta}{\alpha}(1 - \exp(-\alpha(t-s_0-\epsilon))). \tag{7.13}$$

Noting that $a(s)$ is continuous and $\|a(s_0)\|_H = 0$, we find, upon taking the limit $\epsilon \to 0$:

$$\|a(t)\|_H \leq \frac{\beta}{\alpha}(1 - \exp(-\alpha(t-s_0))). \tag{7.14}$$

This implies (7.9). □

We first assume U_0 is smooth. The general case may be treated using an approximation argument. In the proof of Theorem 7.3, we use the above lemma by setting $H = L^2(\Gamma)$. In Sections 8 and 9, we shall also use this lemma by setting $H = \mathbb{R}^N$.

Proof of Theorem 7.3. Suppose the mild solution ceases to exist after some finite time. By an argument similar to the proof of Theorem 6.4, we see that $U(t,x)$ cannot reach the boundary of the invariant cylinder $\mathcal{C}$. By Proposition 6.1, the $C(\Gamma)$ norm of $v(t)$ must must tend to infinity in finite time. We first show that this is not possible.

Consider the variation of constants formula:

$$
\begin{aligned}
&v(t) = I_1 + I_2, \\
&I_1 = \exp(-t\Lambda)v(0),\ I_2 = \int_0^t \exp(-(t-s)\Lambda) f(v,w)ds.
\end{aligned} \tag{7.15}
$$

We first directly estimate the $C(\Gamma)$ norm of $v(t)$. The $C(\Gamma)$ norm of the first term can be estimated as

$$
\|I_1\|_{C(\Gamma)} \le M_c \|v(0)\|_{C(\Gamma)} \le M_c M_0. \tag{7.16}
$$

For I_2, we have:

$$
\|I_2\|_{C(\Gamma)} \le \int_0^t M_c \|f(v,w)\|_{C(\Gamma)} ds \le \int_0^t M_c(M_1 + M_2 \|v(s)\|_{C(\Gamma)})ds. \tag{7.17}
$$

where M_1 and M_2 are constants that depend only on f. We used the Lipschitz continuity of f and the fact that w is bounded ($U(t,x)$ is contained in the invariant cylinder). Using Gronwall's inequality, we have:

$$
\|v(t)\|_{C(\Gamma)} \le C_0 \exp(C_1 t), \tag{7.18}
$$

where C_1 is constants that depend only on f and C_0 depends also on M_0. We thus see that $\|v(t)\|_{C(\Gamma)}$ remains finite and thus the solution is global.

We now show that the $C(\Gamma)$ norm of $v(t)$ is in fact uniformly bounded. For this, we use a standard energy argument. Since U_0 is smooth, U is a classical solution and (2.14a) is satisfied pointwise. First, multiply both sides of (2.14a) with v and integrate in x.

$$
\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(\Gamma)} = -\langle v, \Lambda v\rangle_{L^2(\Gamma)} + \langle f(U), v\rangle_{L^2(\Gamma)} \tag{7.19}
$$

where $\langle \cdot,\cdot\rangle_{L^2(\Gamma)}$ denotes the L^2 inner product. Since Λ is positive semidefinite, we have:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(\Gamma)} &\le \langle f(U), v\rangle_{L^2(\Gamma)} = \langle f_1(U), v\rangle_{L^2(\Gamma)} + \langle f_2(U), v\rangle_{L^2(\Gamma)} \\
&\le -\gamma \|v\|^2_{L^2(\Gamma)} + \langle \eta, v\rangle_{L^2(\Gamma)} \le -\gamma \|v\|^2_{L^2(\Gamma)} + \eta |\Gamma|^{1/2} \|v\|_{L^2(\Gamma)}
\end{aligned} \tag{7.20}
$$

where we used (7.5) in the second inequality and the Cauchy-Schwarz inequality in the last inequality. Here, $|\Gamma|$ denotes the area of Γ. From Lemma 7.4, we see that:

$$
\|v(t)\|_{L^2(\Gamma)} \le M_0 |\Gamma|^{1/2} \exp(-\gamma t) + \frac{\eta}{\gamma}|\Gamma|^{1/2}(1 - \exp(-\gamma t)). \tag{7.21}
$$

where we used $\|v(0)\|_{L^2(\Gamma)} \le |\Gamma|^{1/2}\|v(0)\|_{C(\Gamma)} \le |\Gamma|^{1/2} M_0$. From this, we see that:

$$
\|v(t)\|_{L^2(\Gamma)} \le \frac{2\eta}{\gamma}|\Gamma|^{1/2} \equiv C_2,\ t \ge T_0, \tag{7.22}
$$

where T_0 depends only on M_0.

We now estimate $\|v(t)\|_{H^s(\Gamma)}, 0 < s < 1$ for $t \ge T_0 + T_1$ where $T_1 > 0$ is some fixed number. Consider (7.15) and take the $H^s(\Gamma)$ norm of I_1:

$$
\|I_1\|_{H^s(\Gamma)} \le M_2^{s,0}(1 + T_1^{-s})\|v(t - T_1)\|_{L^2(\Gamma)} \le M_2^{s,0}(1 + T_1^{-s})C_2. \tag{7.23}
$$

where we used (4.73). We now estimate I_2 in the $H^s(\Gamma), 0 < s < 1$ norm.

$$
\|I_2\|_{H^s(\Gamma)} \le \int_{t-T_1}^t M_2^{s,0}(1 + (t-\tau)^{-s})\|f(v,w)\|_{L^2(\Gamma)} d\tau, \tag{7.24}
$$

where we used (4.73). Given that $f(v,w)$ is globally Lipschitz and that w is bounded uniformly in time, we see that:

$$\|f(v,w)\|_{L^2(\Gamma)} \leq C_3,\ t \geq T_0, \tag{7.25}$$

where C_3 is a constant that depends only on C_2. We thus have:

$$\|I_2\|_{H^s(\Gamma)} \leq C_4(T_1 + T_1^{1-s}), \tag{7.26}$$

where C_4 depends only on C_2. Combining estimates for I_1 and I_2 we see that the $\|v(t)\|_{H^s(\Gamma)}, t \geq T_0 + T_1$ is bounded by a constant C_5 that depends only on T_1.

Take $p > 2$ such that $s > 1 - 2/p$ and pick a $\sigma < 1$ so that $\sigma > 2/p$. Note that this is possible since $s > 0$. We now show that the $W_p^\sigma(\Gamma)$ norm of $v(t)$ is uniformly bounded in time for $t \geq T_0 + 2T_1$.

Since $H^s(\Gamma) \hookrightarrow L^p(\Gamma)$ for $s > 1 - 2/p$, we see that $v(t)$ is bounded in $L^p(\Gamma)$ for $t \geq T_0 + T_1$. Hence, $f(v,w)$ is bounded in $L^p(\Gamma)$ by the Lipschitz continuity of f and the L^∞ bound on w. We thus see that:

$$\|v(t)\|_{L^p(\Gamma)} \leq C_6,\ \|f(v,w)\|_{L^p(\Gamma)} \leq C_6,\ t \geq T_0 + T_1 \tag{7.27}$$

where C_6 is a constant that depends only on T_1. Taking the $W_p^\sigma(\Gamma)$ norm in the variation of constants formula (7.15) for $t \geq T_0 + 2T_1$, we have:

$$\begin{aligned} \|v(t)\|_{W_p^\sigma(\Gamma)} &\leq M_p^{\sigma,0}(1 + T_1^{-\sigma}) \|v(t - T_1)\|_{L^p(\Gamma)} \\ &+ \int_{t-T_1}^t M_p^{\sigma,0}(1 + (t-\tau)^{-\sigma}) \|f(u,v)\|_{L^p(\Gamma)} d\tau \\ &\leq M_p^{\sigma,0} C_6 \left(1 + T_1^{-\sigma} + T_1 + \frac{1}{1-\sigma} T_1^{1-\sigma}\right) \end{aligned} \tag{7.28}$$

where we used (4.73) and (7.27). Given $W_p^\sigma(\Gamma) \hookrightarrow C(\Gamma)$ when $\sigma > 2/p$, we see that the $C(\Gamma)$ norm of $v(t)$ is bounded by a constant that depends only on T_1 for $t \geq T_0 + 2T_1 \equiv T$. This proves the second statement in the Theorem. Combining this with (7.18), we see that:

$$\sup_{t \geq 0} \|v(t)\|_{C(\Gamma)} < M \tag{7.29}$$

where M depends only on M_0. □

The Hodgkin-Huxley model has the form:

$$\begin{aligned} f(v, w_1, w_2, w_3) =& - G_{Na} w_1^3 w_2 (v - E_{Na}) - G_K w_3^4 (v - E_K) \\ & - G_L (v - E_L), \end{aligned} \tag{7.30}$$

$$g_k(v, w_k) = \frac{w_{k,\infty}(v) - w_k}{\tau_k(v)}. \tag{7.31}$$

Here, G_{Na}, G_K, G_L are positive constants, E_{Na}, E_K, E_L are constants and $w_{k,\infty}(v)$ and $\tau_k(v)$ are positive functions that depend on v. In particular, $0 < w_{k,\infty}(v) < 1$ for all v.

Corollary 7.5. *The Hodgkin-Huxley model has a global bounded solution for all initial conditions $U_0 \in \mathcal{U}_C \cap X^4$ where $L_k^+ = 1$ and $L_k^- = 0$ in (7.1). For $X = W_p^s(\Gamma), s > 2/p$, the solution is classical. If $U_0 \in \mathcal{U}_{M_0}$, there is a M that depends only on M_0 such that $U(t) \in \mathcal{U}_M$ for all time. There is also a time T that depends only on M_0 such that for $t \geq T$, $U(t) \in \mathcal{U}_{M_*}$ where M_* does not depend on M_0.*

Proof. The invariant cylinder condition (7.2) is satisfied since $\tau_k(v) > 0$ and $0 < w_{k,\infty}(v) < 1$ in (7.31). Condition (7.5) is satisfied if we let:

$$\begin{aligned} f = f_1 + f_2,\ f_1 &= -(G_{Na}w_1^3 w_2 + G_K w_3^4 + G_L)v, \\ f_2 &= (G_{Na}w_1^3 w_2 E_{Na} + G_K w_3^4 E_K + G_L E_L). \end{aligned} \tag{7.32}$$

It is clear that f is globally Lipschitz in the invariant cylinder. The Hodgkin-Huxley model is thus of HH type and we may apply Theorem 7.3 to obtain the desired conclusion. □

We note that many electrophysiology models are of HH type (Morris-Lecar model, cardiac electrophysiology models, see for example, [14]), and are sometimes referred to as conductance-based models.

8. Asymptotic Smoothing and the Global Attractor

In this section, we study the asymptotic behavior of the mild solutions to (2.14). More specifically, we will show that the solutions are "asymptotically smooth", the meaning of which will be specified later. Note that (2.14) does not have an immediate parabolic smoothing property since it is a coupled system of a parabolic equation (2.14a) and a system of ordinary differential equations (2.14b). Once this property is established, we will prove the existence of the global attractor and its finite dimensionality.

We first introduce a structure condition that we shall use in this section and Section 9.2. Let:

$$\mathcal{Y} = \begin{cases} \mathbb{R}^{N+1} & \text{for FN type systems,} \\ \mathcal{C} & \text{for HH type systems,} \end{cases} \tag{8.1}$$

where $\mathcal{C}$ is the invariant cylinder. Consider $g = (g_1, \cdots, g_N)$ in (2.14b). For $y = (y_0, y_w) \in \mathcal{Y}$, let $\frac{\partial g}{\partial y_w}(y)$ denote the $N \times N$ matrix whose (j,k) entry, $1 \le j,k \le N$ is given by $\frac{\partial g_j}{\partial y_k}$. We assume $\frac{\partial g}{\partial y_w}(y)$ satisfies the following property:

$$z \cdot \frac{\partial g}{\partial y_w}(y) z \le -\zeta \left|z\right|^2 \text{ for all } y \in \mathcal{Y}, \text{ and } z \in \mathbb{R}^N \tag{8.2}$$

where $\cdot$ denotes the inner product on $\mathbb{R}^N$ and $|\cdot|$ is the Euclidean distance in $\mathbb{R}^N$. It is easily checked that the above condition for g is satisfied by the FitzHugh-Nagumo and Hodgkin-Huxley systems.

8.1. Asymptotic Smoothing.

In this subsection, we show that the ω-limit set of any bounded solution of (2.14) consists of smooth functions.

Let $U(t), t \ge 0$ be a globally defined mild solution of (2.14) in $(C(\Gamma))^{N+1}$. The *ω-limit set* of the mild solution $U(t)$, which we denote by $\omega(U)$, is defined as follows:

$$\omega(U) = \bigcap_{t \ge 0} \overline{\bigcup_{s \ge t} U(s)}, \tag{8.3}$$

where the overline denotes closure in $(C(\Gamma))^{N+1}$. The following is an elementary characterization of $\omega(U)$:

$$U^* \in \omega(U) \Leftrightarrow \exists t_0 < t_1 < t_2 \to \infty \text{ such that } U(t_k) \to U^* \text{ in } (C(\Gamma))^{N+1}. \tag{8.4}$$

We say $\omega(U)$ attracts $U(t)$ if, for any neighborhood $\mathcal{N}$ of $\omega(U)$ in $(C(\Gamma))^{N+1}$, there is a $T \ge 0$ such that for $t \ge T, U(t) \in \mathcal{N}$. The main result of this subsection is the following:

Theorem 8.1. *Suppose that the system* (2.14) *is of FN or HH-type and that* g *in* (2.14b) *satisfies* (8.2). *If the system is of HH-type, we assume that the initial data is given in the invariant cylinder. Then, any mild solution* $U(t), t \geq 0$ *has a compact* ω*-limit set* $\omega(U)$ *in the* $(C(\Gamma))^{N+1}$ *topology. The set* $\omega(U)$ *attracts* $U(t)$ *and is contained in* $(C^\infty(\Gamma))^{N+1}$.

The above theorem will be proved in several steps. We first prove the following weaker regularity result. We will then use a bootstrap argument to increase regularity.

Proposition 8.2. *Suppose* g *in* (2.14b) *satisfies* (8.2). *Let* $U(t) = (v(t), w(t)) = (v(t), w_1(t), \cdots, w_N(t))$ *be a bounded mild solution to* (2.14) *in* $(C(\Gamma))^{N+1}$, *and let* $\|U(t)\|_{(C(\Gamma))^{N+1}} \leq M$. *We assume that* $U(t,x),\ t \geq 0,\ x \in \Gamma$ *always lies in* $\mathcal{Y}$, *the set on which* (8.2) *is valid. Then,* $\omega(U)$ *is a non-empty compact set in* $(C^\alpha(\Gamma))^{N+1}, 0 < \alpha < 1$ *and attracts* $U(t)$. *Furthermore,*

$$\sup_{U^* \in \omega(U)} \|U^*\|_{(C^\alpha(\Gamma))^{N+1}} \leq K_\alpha, \tag{8.5}$$

where K_α *depends only on* α *and the quantity:*

$$K_0 = \limsup_{t \to \infty} \|U(t)\|_{(C(\Gamma))^{N+1}} = \sup_{U^* \in \omega(U)} \|U^*\|_{(C(\Gamma))^{N+1}}. \tag{8.6}$$

As we stated earlier, equation (2.14) is only partially parabolic. Therefore, we cannot expect immediate smoothing of solutions. Nonetheless, the above proposition shows that every bounded solution is attracted to a subset of $(C^\alpha(\Gamma))^{N+1}$. In this sense, solutions are *asymptotically smooth*.

To prove the above proposition, we need the following lemma:

Lemma 8.3. *Let* g *in* (2.14b) *satisfy* (8.2), *and the mild solution* $U(t)$ *of* (2.14) *satisfy the hypotheses of Proposition 8.2. Given any sequence* $0 \leq t_1 < t_2 < \cdots \to \infty$, *the sequence* $U(t_k), k = 1, 2, \cdots$, *has a convergent subsequence in* $(C(\Gamma))^{N+1}$. *Furthermore, the limit function* $U_\infty = (v_\infty, w_\infty)$ *belongs to* $(C^\alpha(\Gamma))^{N+1}$ *for any* $0 < \alpha < 1$. *The norm* $\|U_\infty\|_{(C^\alpha(\Gamma))^{N+1}}$ *is bounded by a constant that depends only on* M.

Proof. Take some $T > 0$. The function $v(t), t \geq T$ satisfies the following:

$$v(t) = \exp(-T\Lambda)v(t-T) + \int_{t-T}^{t} \exp(-(t-\tau)\Lambda) f(U(\tau))d\tau. \tag{8.7}$$

Take the $W_p^s(\Gamma), 0 < s < 1$ norm of the above:

$$\begin{aligned} \|v(t)\|_{W_p^s(\Gamma)} &\leq M_p^{s,0}(1 + T^{-s}) \|v(t-T)\|_{L^p(\Gamma)} \\ &\quad + \int_{t-T}^{t} M_p^{s,0}(1 + (t-\tau)^{-s}) \|f(U(\tau))\|_{L^p(\Gamma)} d\tau \leq C_0(s, p, T, M) \end{aligned} \tag{8.8}$$

where C_0 is a constant that depends only on s, p, T and M. In the first inequality we used (4.73). In the second inequality we used the fact that $U(t), t \geq 0$ is bounded uniformly in $(C(\Gamma))^{N+1}$ and hence that both $v(t), f(U(t)), t \geq 0$ are bounded uniformly in $L^p(\Gamma)$. Since the above is true for any $0 < s < 1$ and $1 < p < \infty$, this implies a bound on the α-Hölder norm by Sobolev embedding:

$$\|v(t)\|_{C^\alpha(\Gamma)} \leq M_\alpha(\alpha, T, M),\ 0 < \alpha < 1,\ t \geq T, \tag{8.9}$$

where M_α depends only on α, T and M. Since $C^\alpha(\Gamma)$ is compactly embedded in $C(\Gamma)$ we see that $v(t_k), t_k \geq T$ has a convergent subsequence in $C(\Gamma)$.

It now suffices to show that $w(t_k) = (w_1(t_k), \ldots, w_N(t_k)), k = 1, 2, \cdots$ has a convergence subsequence in $(C(\Gamma))^N$. By Arzelà-Ascoli, we must show that the set $w(t_k), k = 1, 2, \cdots$ is equicontinuous. For any $x, x' \in \Gamma$, we have:

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\left|w(x) - w(x')\right|^2\right) &= I_1 + I_2, \\ I_1 &= (w(x) - w(x')) \cdot (g(v(x), w(x)) - g(v(x), w(x'))), \\ I_2 &= (w(x) - w(x')) \cdot (g(v(x), w(x')) - g(v(x'), w(x'))). \end{aligned} \tag{8.10}$$

where we have used the shorthand $w(x) = w(t, x)$ and $w(x') = w(t, x')$. The term I_1 can be estimated as follows:

$$\begin{aligned} I_1 &= (w(x) - w(x')) \cdot G_w(w(x) - w(x')) \leq -\zeta \left|w(x) - w(x')\right|^2 \\ G_w &= \int_0^1 \frac{\partial g}{\partial y_w}(v(x), sw(x) + (1-s)w(x'))ds. \end{aligned} \tag{8.11}$$

where we used (8.2) in the above inequality. For I_2, we have:

$$\begin{aligned} I_2 &\leq \left|w(x) - w(x')\right| \left|g(v(x), w(x')) - g(v(x'), w(x'))\right| \\ &= \left|w(x) - w(x')\right| \left|G_v\right| \left|(v(x) - v(x'))\right| \\ G_v &= \int_0^1 \frac{\partial g}{\partial y_0}(sv(x) + (1-s)v(x'), w(x'))ds. \end{aligned} \tag{8.12}$$

We see that $|G_v| \leq \eta$ where the constant η depends only on M. Using (8.9) to bound $|(v(x) - v(x'))|$, we have:

$$I_2 \leq \eta M_\alpha \left|x - x'\right|^\alpha \left|w(x) - w(x')\right|, \tag{8.13}$$

for $t \geq T$. Combining (8.10), (8.11) and (8.13), and using Lemma 7.4, we find:

$$\left|w(t, x) - w(t, x')\right| \leq 2M \exp(-\zeta(t - T)) + \eta M_\alpha \zeta^{-1} \left|x - x'\right|^\alpha. \tag{8.14}$$

for $t \geq T$.

We are now ready to show equicontinuity of the set $w(t_k), k = 1, 2, \cdots$. Take an arbitrary $\epsilon > 0$. Take $\delta_1 > 0$ small enough so that

$$\eta M_\alpha \zeta^{-1} \left|x - x'\right|^\alpha \leq \frac{\epsilon}{2} \tag{8.15}$$

whenever $|x - x'| \leq \delta_1$. Take $T_1 > T$ large enough so that $2M \exp(-\zeta(T_1 - T)) \leq \epsilon/2$. We see from (8.14) that $|w(t, x) - w(t, x')| \leq \epsilon$ for $|x - x'| \leq \delta_1$ whenever $t_k \geq T_1$. Take $\delta_2 > 0$ small enough so that $|w(t_k, x) - w(t_k, x')| \leq \epsilon$ for $t_k < T_1$. This is possible because there are only finitely many t_k smaller than T_1. By taking $\delta = \min(\delta_1, \delta_2)$, we see that $|w(t_k, x) - w(t_k, x')| \leq \epsilon$ if $|x - x'| \leq \delta$.

Extract a convergent subsequence from $U(t_k)$, which we shall still denote by $U(t_k) = (v(t_k), w(t_k))$. Let $U_\infty = (v_\infty, w_\infty)$ be the limiting function. Since the $C^\alpha(\Gamma)$ norm of $v(t_k)$ is bounded, it is clear that $v_\infty \in C^\alpha(\Gamma)$. Substituting $t = t_k$ in (8.14) and letting $k \to \infty$, we conclude that:

$$\left|w_\infty(x) - w_\infty(x')\right| \leq \eta M_\alpha \zeta^{-1} \left|x - x'\right|^\alpha. \tag{8.16}$$

This shows that $w_\infty \in (C^\alpha(\Gamma))^{N+1}$. Since M_α depends only on M (see (8.9)), we see that $\|U_\infty\|_{(C^\alpha(\Gamma))^{N+1}}$ is bounded by a constant that depends only on M. $\square$

We may now use the above lemma to prove Proposition 8.2.

Proof of Proposition 8.2. The former part of Lemma 8.3 implies that the set $\gamma^+(U) = \bigcup_{t\geq 0} U(t)$ is a relatively compact set in $(C(\Gamma))^{N+1}$. Therefore, $\omega(U)$ is a non-empty compact set in $(C(\Gamma))^{N+1}$. It is also well-known that the relative compactness of the $\gamma^+(U)$ implies that $\omega(U)$ attracts $U(t)$ (see, for example, [33], [11] for a proof). Given the characterization (8.4) of $\omega(U)$, it is immediate from Lemma 8.3 that $\omega(U)$ is contained in $(C^\alpha(\Gamma))^{N+1}$ and that K_α in (8.5) depends only on $\sup_{t\geq 0} \|U(t)\|_{C(\Gamma)^{N+1}}$. Consider the ω-limit set of the mild solution $U_s(t)$ such that $U_s(0) = U(s), s > 0$. Clearly, $\omega(U) = \omega(U_s)$. Therefore, K_α may be taken to depend only on $\sup_{t\geq s} \|U(t)\|_{C(\Gamma)^{N+1}}$, and by taking the limit $s \to \infty$, we obtain (8.5) with K_α depending only on (8.6). Given the compact embedding $(C^\alpha(\Gamma))^{N+1} \hookrightarrow (C^\beta(\Gamma))^{N+1}$ for $0 < \beta < \alpha < 1$, we see that $\omega(U)$ is a compact subset of $(C^\alpha(\Gamma))^{N+1}, 0 < \alpha < 1$. □

Next we improve the regularity of the ω-limit set by a bootstrap argument. More precisely, we have the following lemma:

Lemma 8.4. *Under the hypotheses of Proposition 8.2, $\omega(U)$ is contained in $(C^{1+\alpha}(\Gamma))^{N+1}, 0 < \alpha < 1$. Moreover,*

$$\sup_{U^*\in\omega(U)} \|U^*\|_{(C^{1+\alpha}(\Gamma))^{N+1}} \leq K_{1+\alpha}, \tag{8.17}$$

where $K_{1+\alpha}$ depends only on α and (8.6).

Proof. Take any point in $U^* = (v^*, w^*) \in \omega(U)$ and let $V(t) = (v(t), w(t))$ be the mild solution to (2.14) so that $V(0) = U^*$. It is easy to see that the set $\bigcup_{t\geq 0} V(t)$ is contained in $\omega(U)$. Furthermore, $V(t)$ possesses a negative extension so that $V(t), t \in \mathbb{R}$ is a mild solution of (2.14) and is contained in $\omega(U)$ (see, for example, [33] or [11] for a proof of this well-known fact). Since $V(t)$ is contained in $\omega(U)$, $V(t) \in (C^\alpha(\Gamma))^{N+1}$ by Lemma 8.2. Therefore, $V(t) \in (W^s_p(\Gamma))^{N+1}, 0 < s < 1, 1 < p < \infty$ by Sobolev embedding. This implies that $V(t)$ is a mild solution of (2.14) in $(W^s_p(\Gamma))^{N+1}$, and by Theorem 5.5, $V(t)$ is a classical solution in $(W^s_p(\Gamma))^{N+1}$. In particular, $v(t)$ belongs to $C([a, b]; W^{s+1}_p(\Gamma))$ for any $-\infty < a < b < \infty$. We have, by the embedding $W^{s+1}_p(\Gamma) \hookrightarrow C^{1+\alpha}(\Gamma), \alpha < s - 2/p$,

$$v(t) \in C([a, b]; C^{1+\alpha}(\Gamma)),\ 0 < \alpha < 1,\ \text{for any}\ -\infty < a < b < \infty. \tag{8.18}$$

Now, consider:

$$v^* = \exp(-T\Lambda)v(-T) + \int_{-T}^{0} \exp(s\Lambda)f(V(s))ds. \tag{8.19}$$

for $T \geq 0$. By (8.5) of Corollary 8.2, we see that $\|V(s)\|_{C^\alpha(\Gamma)}$, and hence $\|f(V(s))\|_{C^\alpha(\Gamma)}$ is bounded by a constant that depends only on M_ω defined in (8.6). Since $C^\alpha(\Gamma) \hookrightarrow W^s_p(\Gamma)$ for $\alpha > s-2/p$, by an argument similar to (8.8), we see that $\|v^*\|_{W^s_p(\Gamma)}, 1 \leq s < 2, 1 < p < \infty$, and is bounded by a constant that does not depend on the choice of U^* in $\omega(U)$. By the embedding $W^s_p(\Gamma) \hookrightarrow C^{1+\alpha}(\Gamma), 0 < \alpha < s - 1 - 2/p$, we have:

$$\sup_{U^*\in\omega(U)} \|v^*\|_{C^{1+\alpha}(\Gamma)} \leq M^v_{1+\alpha} \tag{8.20}$$

where $M^v_{1+\alpha}$ depends only on (8.6). This implies in particular that $v(t)$ is a bounded continuous function on $\mathbb{R}$ with values in $C^{1+\alpha}(\Gamma)$, a statement that is stronger than (8.18).

Introduce a finite coordinate cover of Γ. Take one coordinate patch in this cover, and identify this with the corresponding open set $\mathcal{D} \subset \mathbb{R}^2$. Let the coordinates on $\mathcal{D}$ be (x_1, x_2) and $e_i, i = 1, 2$ be the two coordinate vectors. To show that $w \in (C^{1+\alpha}(\Gamma))^N$ it is sufficient to show that the following quantity is bounded:

$$\sup_{x',x\in\mathcal{D}} \left|\frac{\partial w}{\partial x_i}(x') - \frac{\partial w}{\partial x_i}(x)\right| |x'-x|^{-\alpha}, \ 0 < \alpha < 1, \ i = 1, 2. \tag{8.21}$$

We first show that $\frac{\partial w}{\partial x_i}$ is a continuous function on $\mathcal{D}$ and subsequently show that the above quantity is bounded.

Consider the difference

$$w(t, x') - w(t, x), \ x' = x + re_i, \ i = 1, 2, \tag{8.22}$$

where $|r|$ is taken small enough so that both x and $x + re_i$ lies within $\mathcal{D}$. Since $v(t), w(t)$ are continuous functions of x, (2.14b) is satisfied pointwise. The difference (8.22) satisfies:

$$\begin{aligned}
\frac{\partial}{\partial t}(w(t, x') - w(t, x)) &= g(v(t, x'), w(t, x')) - g(v(t, x), w(t, x)) \\
&= P(w(t, x') - w(t, x)) + Q, \\
P(t, x, x') &= \int_0^1 \frac{\partial g}{\partial y_w}(v(t, x), (1-s)w(t, x) + sw(t, x'))ds, \\
Q(t, x, x') &= g(v(t, x'), w(t, x')) - g(v(t, x), w(t, x')),
\end{aligned} \tag{8.23}$$

We may solve the above system for $w(t, x') - w(t, x)$ to find:

$$\begin{aligned}
w^*(x') - w^*(x) = \exp&\left(\int_{-T}^0 P(s, x, x')ds\right)(w(-T, x') - w(-T, x)) \\
&+ \int_{-T}^0 \exp\left(\int_t^0 P(s, x, x')ds\right) Q(t, x, x')dt,
\end{aligned} \tag{8.24}$$

where $T \geq 0$. Consider the expression:

$$z(t) = \exp\left(\int_t^0 P(s, x, x')ds\right) z_0, \tag{8.25}$$

where $t \leq 0$ and $z(0) = z_0$ is a vector in $\mathbb{R}^N$. The function $z(t)$ satisfies the equation $\frac{d}{dt}z = -P(t)z$. Therefore, we have:

$$\frac{1}{2}\frac{d}{dt}|z|^2 \geq \zeta |z|^2 \tag{8.26}$$

by (8.23) and (8.2). Integrating this differential inequality, we find:

$$|z(t)| \leq \exp(\zeta t)|z_0|, \ t \leq 0. \tag{8.27}$$

We see, therefore, that:

$$\left|\exp\left(\int_t^0 P(s, x, x')ds\right)\right| \leq \exp(\zeta t), \ t \leq 0. \tag{8.28}$$

where the absolute value sign above denotes the Euclidean matrix norm. Let us now consider limit $T \to \infty$ in (8.24). By (8.28) and the fact that $w(t) \in \omega(U)$ is bounded in $(C(\Gamma))^N$, the first term in (8.24) tends to 0 as $T \to \infty$. Thus,

$$w^*(x') - w^*(x) = \int_{-\infty}^{0} \exp\left(\int_t^0 P(s,x,x')ds\right) Q(t,x,x')dt. \tag{8.29}$$

Noting that $x' = x + re_i$, divide both sides of the above by r and consider the limit as $r \to 0$. For the integrand, we have:

$$\lim_{r\to 0} \frac{1}{r} \exp\left(\int_t^0 P(s,x,x')ds\right) Q(t,x,x') = \exp\left(\int_t^0 \frac{\partial g}{\partial y_w} ds\right) \frac{\partial g}{\partial y_0}\frac{\partial v}{\partial x_i} \tag{8.30}$$

where we used the fact that $v(t) \in C^1(\Gamma)$ (and hence $v(t) \in C^1(\mathcal{D})$). Note that

$$\frac{1}{r}Q(t,x,x') = \left(\int_0^1 \frac{\partial g}{\partial y_0}(sv(t,x) + (1-s)v(t,x'), w(t,x'))ds\right) \frac{\partial v}{\partial x_i}(t, x + r\theta e_i), \tag{8.31}$$

for some $0 < \theta < 1$. Given (8.5) and (8.20), Q/r is bounded by a constant M_Q independent of x, x' or t. Thus,

$$\left|\frac{1}{r} \exp\left(\int_t^0 P(s,x,x')ds\right) Q(t,x,x')\right| \le \exp(\zeta t) M_Q. \tag{8.32}$$

where we used (8.28). The right hand side of the above is clearly integrable over $-\infty < t \le 0$, and, therefore, by the Lebesgue Dominated Convergence theorem, we conclude that:

$$\frac{\partial w^*}{\partial x_i} = \int_{-\infty}^{0} \left(\exp\left(\int_t^0 \frac{\partial g}{\partial y_w} ds\right) \frac{\partial g}{\partial y_0}\frac{\partial v}{\partial x_i}\right) dt. \tag{8.33}$$

It is easy to check (using the Lebesgue Dominated Convergence Theorem) that the right hand side is a continuous function of x. Therefore, w^* belongs to $C^1(\mathcal{D})$. It is also clear from (8.32) that

$$\sup_{U^* \in \omega(U)} \|w^*\|_{C^1(\mathcal{D})} \le M_1^w \tag{8.34}$$

where M_1^w is a constant that depends only on (8.6). Given that the above is satisfied on each of the finite number of coordinate covers, we may replace $C^1(\mathcal{D})$ with $C^1(\Gamma)$ by replacing M_1^w with a larger value if need be.

By calculations similar to those leading to (8.33), we can show that:

$$\frac{\partial w}{\partial x_i}(t,x) = \int_{-\infty}^{t} \left(\exp\left(\int_\tau^t \frac{\partial g}{\partial y_w} ds\right) \frac{\partial g}{\partial y_0}\frac{\partial v}{\partial x_i}\right) d\tau. \tag{8.35}$$

Let us take the derivative of the above with respect to t:

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x_i}\right) = \frac{\partial g}{\partial y_w}\frac{\partial w}{\partial x_i} + \frac{\partial g}{\partial y_0}\frac{\partial v}{\partial x_i}. \tag{8.36}$$

In deriving this equality, we used the fact that $v(t,x), w(t,x)$ and $\frac{\partial v}{\partial x_i}(t,x)$ are continuous functions of t and the decay estimate (8.28). The continuity of $\frac{\partial v}{\partial x_i}(t,x)$ with respect to t follows from (8.18).

The rest of the proof is analogous to the latter half of the proof of Lemma 8.3. Estimate the difference $\delta_w = \frac{\partial w}{\partial x_i}(t,x') - \frac{\partial w}{\partial x_i}(t,x)$, where x and x' are in $\mathcal{D}$. An easy calculation yields:

$$\frac{1}{2}\frac{d}{dt}\left|\delta_w\right|^2 = I_1 + I_2, \quad I_1 = \delta_w \cdot \frac{\partial g}{\partial y_w}(x)\delta_w,$$
$$I_2 = \delta_w \cdot \left(\left(\frac{\partial g}{\partial y_w}(x') - \frac{\partial g}{\partial y_w}(x)\right)\frac{\partial w}{\partial x_i}(x') + \left(\frac{\partial g}{\partial y_0}(x')\frac{\partial v}{\partial x_i}(x') - \frac{\partial g}{\partial y_0}(x)\frac{\partial v}{\partial x_i}(x)\right)\right), \tag{8.37}$$

where $\frac{\partial g}{\partial y_w}(x') = \frac{\partial g}{\partial y_w}(v(t,x'), w(t,x'))$ and so on. It is clear from (8.5) and (8.20) that

$$|I_2| \leq K_I^\alpha \left|x - x'\right|^\alpha \left|\delta_w\right|, \; 0 < \alpha < 1, \tag{8.38}$$

where K_I^α depends only on α and (8.6). Using (8.2) for the term I_1 in (8.37), we obtain the following differential inequality:

$$\frac{1}{2}\frac{\partial}{\partial t}\left|\delta_w\right|^2 \leq -\zeta\left|\delta_w\right|^2 + K_I^\alpha \left|x - x'\right|^\alpha \left|\delta_w\right|. \tag{8.39}$$

Solving this differential inequality using Lemma 7.4, and noting that $\frac{\partial w}{\partial x_i}$ is bounded uniformly in t by (8.34), we obtain:

$$\delta_w(0) = \left|\frac{\partial w^*}{\partial x_i}(x') - \frac{\partial w^*}{\partial x_i}(x)\right| \leq \zeta^{-1} K_I^\alpha \left|x - x'\right|^\alpha. \tag{8.40}$$

This implies that:

$$\sup_{U^* \in \omega(U)} \|w^*\|_{(C^{1+\alpha}(\mathcal{D}))^{N+1}} \leq M_{1+\alpha}^w, \tag{8.41}$$

where $M_{1+\alpha}^w$ depends only on α and on (8.6). Since the above is valid on each of the finite number of coordinate covers, we may replace $C^{1+\alpha}(\mathcal{D})$ above with $C^{1+\alpha}(\Gamma)$. □

We now iterate the above argument to obtain higher order regularity of the ω-limit set.

Lemma 8.5 (Higher order asymptotic smoothing)**.** *Under the hypotheses of Proposition 8.2, $\omega(U) \subset (C^\infty(\Gamma))^{N+1}$.*

Proof. The proof is by induction. Each step of the induction is similar to the proof of the previous lemma and we only give a brief description of the proof. Suppose $\omega(U) \in (C^{k+\alpha}(\Gamma))^{N+1}, k \geq 1, 0 < \alpha < 1$. Suppose also that, for any solution $V(t) = (v(t), w(t))$ contained in $\omega(U)$, any k-th order partial derivative $\partial_x^q w$ with respect to x satisfies the differential equation that is obtained by formally applying ∂_x^q to (2.14b):

$$\frac{\partial}{\partial t}\partial_x^q w = \frac{\partial g}{\partial y_w}\partial_x^q w + R. \tag{8.42}$$

Here, R depends only on partial derivatives of w of order up to $k-1$, and on partial derivatives of v of order up to k. For $k = 1$, the above is precisely equation (8.36).

Take an arbitrary point $U^* = (v^*, w^*) \in \omega(U)$ and let $V(t) = (v(t), w(t)) \in \omega(U)$ be a solution that passes through this point. By the induction hypothesis $\omega(U) \subset (C^{k+\alpha}(\Gamma))^{N+1}, k \geq 1$, we immediately see that $v(t) \in C^{k+1+\alpha}(\Gamma)$ by the regularizing properties of $\exp(-t\Lambda)$. We now use (8.42) to estimate the finite difference quotients of $\partial_x^q w$ by solving the linear differential equation satisfied by the difference quotients. We conclude, in the same way as in Lemma 8.4, that any partial derivative $\partial_x^{q'} w$ of order $k+1$ is continuous with respect to x and that it satisfies a differential equation of the form (8.42). We may then estimate the α-Hölder norm of $\partial_x^{q'} w^*$ by the argument used to prove (8.41). □

Theorem 8.1 is now obtained as a direct consequence of the above results.

Proof of Theorem 8.1. The hypotheses of Theorem 6.7 or Theorem 7.3 guarantees that all solutions are bounded in $(C(\Gamma))^{N+1}$. The conclusion follows by application of Proposition 8.2, Lemma 8.4 and Lemma 8.5. □

8.2. **Global Attractor.** We first recall the definition of the global attractor. Let Z be a complete metric space. The one-parameter family of operators $\Phi(t) : Z \to Z, t \geq 0$ is a *semiflow* on Z if:

(1) $\Phi(0)u = u$ for $u \in Z$.
(2) $\Phi(t)\Phi(s)u = \Phi(t+s)u$ for all $u \in Z, t, s \geq 0$.
(3) $\Phi(t)u$ is jointly continuous in $u \in Z$ and $t \geq 0$.

For a set $A \subset Z$, we shall often use the notation $\Phi(t)A$ to denote the set consisting of all points $\Phi(t)u, u \in A$.

We say that $\mathcal{A} \subset Z$ is a *global attractor* of the semiflow $\Phi(t)$ if the following conditions hold.

(1) $\mathcal{A}$ is a non-empty compact set, and is invariant in the following sense:
$$\Phi(t)\mathcal{A} = \mathcal{A},\ t \geq 0. \tag{8.43}$$
(2) Every compact invariant set of the semiflow must be contained in $\mathcal{A}$.
(3) For any bounded set $B \in Z$, $\mathcal{A}$ attracts B. That is to say, given any neighborhood $\mathcal{N}$ of $\mathcal{A}$, there exists a $T \geq 0$ so that $\Phi(t)B \subset \mathcal{N}$ for $t \geq T$.

The second condition is in fact a consequence of the first and the third (see Lemma 23.3 and discussion on p.55 of [33]).

In what follows, $\Phi(t)$ will denote the semiflow defined by the mild solutions of (2.14) where we set:
$$Z = \begin{cases} Y = (C(\Gamma))^{N+1} & \text{for FN-type systems} \\ \mathcal{U}_\mathcal{C} & \text{for HH-type systems.} \end{cases} \tag{8.44}$$
Here, $\mathcal{U}_\mathcal{C} \subset (C(\Gamma))^{N+1}$ is as defined in (7.6). The first two defining properties of semiflows are obviously satisfied. The third property follows from the continuous dependence of mild solutions on initial data.

We first have the following lemma.

Lemma 8.6. *Let g in* (2.14b) *satisfy* (8.2). *Suppose $U_k, k = 1, 2, \cdots$ are mild solutions to* (2.14) *and satisfy*
$$\|U_k(t)\|_{(C(\Gamma))^{N+1}} \leq M,\ k = 1, 2, \cdots,\ t \geq 0. \tag{8.45}$$
For any sequence $0 \leq t_1 < t_2 \cdots \to \infty$, $U_k(t_k)$ has a convergent subsequence in $(C(\Gamma))^{N+1}$. Furthermore, the limit function $U_\infty = (v_\infty, w_\infty)$ belongs to $(C^\alpha(\Gamma))^{N+1}$ and $\|U_\infty\|_{(C^\alpha(\Gamma))N+1}$ is bounded by a constant that depends only on M.

Proof. The proof is almost identical to that of Lemma 8.3. Let $U_k(t) = (v^k(t), w^k(t)) = (v^k(t), w_1^k, \cdots, w_N^k(t))$. Pick a $T > 0$. For $t \geq T$, $\left\|v^k(t)\right\|_{W_p^s(\Gamma)}, 0 < s < 1, p \geq 2$ is bounded by a constant that depends only on M. This follows from a calculation identical to (8.8). By the embedding $W_p^s(\Gamma) \hookrightarrow C^\alpha(\Gamma)$ for $0 < \alpha < s - 2/p$, we see that $\left\|v^k(t_k)\right\|_{C^\alpha(\Gamma)}, t_k \geq T$ is bounded by a constant that depends only on M and hence forms a precompact set.

For $w^k(t), t \geq T$, we can obtain the following bound analogously to (8.14):

$$\left|w^k(t,x) - w^k(t,x')\right| \leq 2M \exp(-\zeta(t-T)) + M_\alpha \left|x - x'\right|^\alpha, \; x, x' \in \Gamma, \qquad (8.46)$$

where $0 < \alpha < 1$ and M_α is a constant that depends only on M. From this bound, we may argue as in Lemma 8.3 to conclude that the set $w^k(t_k), k = 1, 2, \cdots \to \infty$ is equicontinuous in $(C(\Gamma))^{N+1}$.

The last assertion on the α-Hölder norm of the limit function follows from the same argument as in Lemma 8.3. □

Theorem 8.7. *Suppose system* (2.14) *is either of FN or HH-type and that g in* (2.14b) *satisfies* (8.2). *Then, the semiflow defined by the mild solutions of* (2.14) *possesses a global attractor $\mathcal{A}$. The set $\mathcal{A}$ is a subset of* $(C^\infty(\Gamma))^{N+1}$.

Proof. The conclusions of the theorem follow from from the general theory of global attractors that are dissipative and asymptotically smooth in the sense of [11] or asymptotically compact in the sense of [33]. However, we derive the results directly from the above estimates for the convenience of the reader.

If (2.14) is of FN-type, let,

$$B_{M_*} = \{U \in Z | \, \|v\|_{C(\Gamma)} \leq M_*, \|w_k\|_{C(\Gamma)} \leq B_k(M_*), k = 1, \cdots, N\} \qquad (8.47)$$

where $U = (u, w_1, \cdots, w_N)$ and M_* and B_k are as in Theorem 6.7. If (2.14) is of HH-type, let,

$$B_{M_*} = \mathcal{U}_{M_*} \qquad (8.48)$$

where $\mathcal{U}_M$ is as in (7.7) and M_* is as in Theorem 7.3.

Take any sequence of points $U_k \in B_{M_*}, k = 1, 2, \cdots$ and a sequence of times $0 \geq t_1 < t_2 < \cdots \to \infty$. Consider the sequence $\Phi(t_k)U_k, k = 1, 2, \cdots$. If (2.14) is of FN-type, $\Phi(t_k)U_k$ stays within B_{M_*}. If (2.14) is of HH-type, $\Phi(t_k)U_k$ stays within $\mathcal{U}_M$ where M is a constant that depends only on M_*. In either case, $\|\Phi(t_k)U_k\|_{(C(\Gamma))^{N+1}}$ is bounded by a constant that depends only on M_*. From Lemma 8.6, the sequence $\Phi(t_k)U_k, k = 1, 2, \cdots$ has a convergent subsequence. In other words, B_{M_*} is *asymptotically compact*. Define the ω-limit set of B_{M_*} as follows:

$$\omega(B_{M_*}) = \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} \Phi(t) B_{M_*}}. \qquad (8.49)$$

where the over line denotes closure in $(C(\Gamma))^{N+1}$. It is easy to show that this set coincides with the set of all limit points of convergent sequences of the form $\Phi(t_k)U_k, k = 1, 2, \cdots$ (compare (8.4)). The asymptotic compactness of B_{M_*} implies that $\omega(B_{M_*})$ is a non-empty compact set, and that $\omega(B_{M_*})$ attracts B_{M_*} (see Lemma 23.1 of [33]). Note also that $\omega(B_{M_*})$ is invariant with respect $\Phi(t)$. This is a general property of ω-limit sets.

Now, consider any bounded set $B \in Z$. By either Theorem 6.7 or Theorem 7.3, $\Phi(t)B \subset B_{M_*}$ for $t \geq T_B$ where $T_B > 0$ depends only on B. Since B_{M_*} is attracted to $\omega(B_{M_*})$, B is also attracted to $\omega(B_{M_*})$. We thus see that $\mathcal{A} = \omega(B_{M_*})$ is the global attractor of the semiflow $\Phi(t)$.

Since $\mathcal{A}$ consists of limit points of convergent sequences of the form $\Phi(t_k)U_k, k = 1, 2, \cdots$, we see from Lemma 8.6 that $\mathcal{A}$ is a bounded set in $(C^\alpha(\Gamma))^{N+1}, 0 < \alpha < 1$.

That $\mathcal{A} \subset (C^\infty(\Gamma))^{N+1}$ now follows from an argument identical to Lemma 8.4 and Lemma 8.5. □

We now show that the attractor $\mathcal{A}$ is finite dimensional. We first recall some definitions. Let K be a compact subset of a metric space X. Define $N(r, K)$ to be the smallest integer N such that K can be covered by N balls of radius r. The *limit capacity* (or *fractal dimension*) of K is defined by:

$$d_C(K) = \limsup_{r \to 0} \frac{\log N(r, k)}{\log(1/r)}. \tag{8.50}$$

We next define the Hausdorff dimension. Let $B(x, r)$ be the ball of radius $r > 0$ centered at $x \in X$. Consider the covering:

$$K \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), \ r_i \leq r. \tag{8.51}$$

Let

$$\mu_H^{\alpha,r}(K) = \inf\left(\sum_{i=1}^{\infty} r_i^\alpha\right), \ \alpha > 0, \tag{8.52}$$

where the infimum is taken over all countable coverings that satisfy (8.51). Then define:

$$\mu_H^\alpha(K) = \lim_{r \to 0} \mu_H^{\alpha,r}(K). \tag{8.53}$$

$\mu_H^{\alpha,r}(K)$ is clearly non-negative and is decreasing as $r \to 0$, and therefore, the above limit exists. It is not difficult to show, that if $\mu_H^\alpha(K) = 0$, then $\mu_H^\beta(K) = 0$ for any $\beta \geq \alpha$. The *Hausdorff dimension* $d_H(K)$ is defined by:

$$d_H(K) = \inf\{\alpha > 0 | \mu_H^\alpha(K) = 0\}. \tag{8.54}$$

It is easy to show that $d_H(K) \leq d_C(K)$.

To show that $\mathcal{A}$ is finite dimensional, we shall use the following result due to [20] in the case of separable Hilbert spaces which was subsequently generalized to Banach spaces in [21].

Let $\mathcal{O}$ be an open subset of a Banach space X. Let Ψ be a C^1 map from $\mathcal{O}$ to $\mathcal{O}$ that is C^1 in the sense that it has continuous Fréchet derivatives L_Ψ. That is to say, for each $x \in \mathcal{O}$, $L_\Psi(x)$ is a bounded operator from X to itself and depends continuously on x.

Proposition 8.8 ([20], [21]). *Let Ψ be a C^1 map defined on an open subset $\mathcal{O}$ of a Banach space X. Let $J \subset \mathcal{O}$ be a compact set such that $J \subset \Psi(J)$. For any point $x \in J$, suppose the derivative of Ψ, $L_\Psi(x)$, can be written as:*

$$L_\Psi(x) = L_1 + L_2, \tag{8.55}$$

where L_1 is a contraction (operator norm is smaller than 1) and L_2 is compact. Then J has finite limit capacity and thus finite Hausdorff dimension.

We shall apply the Proposition 8.8 by taking $\Psi = \Phi(T)$ where $T > 0$, $J = \mathcal{A}$ and $\mathcal{O} = Z^\circ$ where Z° denotes the interior of Z. For FN type systems, $Z^\circ = Z$ (see (8.44)).

In applying Proposition 8.8, it is essential that the existence of decomposition (8.55) is required only on the global attractor $\mathcal{A}$. Indeed, such a decomposition may not be possible on the whole space Z° since (2.14) does not have an immediate smoothing property.

Let us first calculate the derivative of $\Phi(t)$. Let $U^*(t) = \Phi(t)U_0^*, t \geq 0$. The linearization of system (2.14) around the solution $U^*(t)$ takes the following form:

$$\frac{\partial v}{\partial t} = -\Lambda v + \frac{\partial f}{\partial y_0}(U^*(t))v + \frac{\partial g}{\partial y_w}(U^*(t))w, \tag{8.56a}$$

$$\frac{\partial w}{\partial t} = \frac{\partial g}{\partial y_0}(U^*(t))v + \frac{\partial g}{\partial y_w}(U^*(t))w. \tag{8.56b}$$

Given that $U^*(t)$ is bounded in $Y = (C(\Gamma))^{N+1}$, it is standard that the above linearized system has global mild solutions in $Y = (C(\Gamma))^{N+1}$ for any initial data in Y. Define:

$$\Phi_L(U_0^*, t) : U_0 = (v_0, w_0) \in Y \mapsto (v(t), w(t)) \in Y, \tag{8.57}$$

where $(v(t), w(t))$ is the mild solution to (8.56) with initial data (v_0, w_0). It is easily seen that $\Phi_L(U_0^*, t)$ is a bounded linear map from Y to Y.

Lemma 8.9. *The map $\Phi(t), t \geq 0$, for each fixed t, is a C^1 map in Z°. The Fréchet derivative of $\Phi(t)$ at any point $U_0^* \in Z^\circ$ is given by $\Phi_L(U_0^*, t)$.*

Proof. This is a standard result. See for example Theorem 49.2 of [33]. □

We are now ready to prove the following.

Theorem 8.10. *The global attractor $\mathcal{A}$ has finite limit capacity and hence finite Hausdorff dimension.*

Proof. For any $T > 0$, we show that the map $\Phi(T) : Z^\circ \to Z^\circ$ and the set $\mathcal{A} \subset Z^\circ$ satisfy the hypotheses of Proposition 8.8. The set $\mathcal{A}$ is a global attractor and hence is compact and invariant under the map $\Phi(T)$. From the above Lemma, $\Phi(T)$ is a C^1 map from Z° to itself. We have thus only to check that the derivative at each point in $\mathcal{A}$ admits a decomposition of the form (8.55).

Take any point $U_0^* \in \mathcal{A}$. Let $\Phi(t)U_0^* = U^*(t), t \geq 0$. For any $U_0 = (v_0, w_0) \in Y$, let $\Phi_L(U_0^*, t)U_0 = (v(t), w(t))$. Define $L_1(t) : (v_0, w_0) \mapsto (0, w_1(t))$ so that $w_1(t)$ is the solution to the equation:

$$\frac{\partial w_1}{\partial t} = \frac{\partial g}{\partial y_w}(U^*(t))w_1(t),\ w_1(0) = w_0. \tag{8.58}$$

Given (8.2), it is clear that $L_1(t)$ is a contraction on $(C(\Gamma))^{N+1}$ for any $t > 0$.

We now show that the map $L_2(T) \equiv \Phi(U_0^*, T) - L_1(T)$ is compact. We first show that the v-component of $L_2(T)$ is compact. Let $L_2(t) : (v_0, w_0) \mapsto (v(t), w_2(t)) = (v(t), w(t) - w_1(t))$. By Lemma 8.9 and equation (8.56), $v(t)$ satisfies:

$$v(t) = \exp(-t\Lambda)v_0 + \int_0^t \exp(-(t-s)\Lambda)\left(\frac{\partial f}{\partial y_0}(U^*(s))v(s) + \frac{\partial g}{\partial y_w}(U^*(s))w(s)\right) ds \tag{8.59}$$

First, note that U^* is bounded in Y. We also have:

$$\|\Phi_L(U_0^*, s)\|_{\mathcal{L}(Y)} \leq M_\Phi,\ 0 \leq s \leq T, \tag{8.60}$$

where the above denotes the operator norm of linear maps from Y to itself and M_ϕ is a constant that depends only on T. We thus see that:

$$\left\| \frac{\partial f}{\partial y_0}(U^*(s))v(s) + \frac{\partial g}{\partial y_w}(U^*(s))w(s) \right\| \leq M_1 \|U_0\|_Y,\ 0 \leq s \leq T, \tag{8.61}$$

where M_1 is a constant that depends only on t. We may now estimate the $C^\alpha(\Gamma), \alpha > 0$ norm of $v(t), 0 \le t \le T$ by first estimating the $W_p^\sigma(\Gamma), \sigma - 2/p > \alpha$ norm and then using embedding, similarly to (8.8) and (8.9). We have:

$$\|v(t)\|_{C^\alpha(\Gamma)} \le M_2(1+t^{-\beta}) \|U_0\|_Y, \ 0 \le t \le T, \tag{8.62}$$

where $\alpha < \beta < 1$ and M_2 depends on T. This shows that the v-component of $L_2(T)$ is compact.

We now prove that the w-component of $L_2(T)$ is compact. In showing this, we will use the fact that $\mathcal{A} \subset C^\alpha(\Gamma)^{N+1}$. At each point $x \in \Gamma$, $w_2(t,x)$ satisfies:

$$\frac{\partial w_2}{\partial t} = \frac{\partial g}{\partial y_0}(U^*(t))v + \frac{\partial g}{\partial y_w}(U^*(t))w_2, \ w_2(0) = 0. \tag{8.63}$$

Take any two points $x, x' \in \Gamma$. We have:

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}|\delta_w|^2 = I_1 + I_2, \ \delta_w = w_2(t,x') - w_2(t,x), \\ &I_1 = \delta_w \cdot \left(\frac{\partial g}{\partial y_0}(U^*(t,x'))v(t,x') - \frac{\partial g}{\partial y_0}(U^*(t,x))v(t,x)\right), \\ &I_2 = \delta_w \cdot \left(\frac{\partial g}{\partial y_w}(U^*(t,x'))w_2(t,x') - \frac{\partial g}{\partial y_w}(U^*(t,x))w_2(t,x)\right). \end{aligned} \tag{8.64}$$

We estimate I_1 as follows:

$$\begin{aligned} |I_1| &\le |\delta_w| \left|\frac{\partial g}{\partial y_0}(U^*(t,x'))\right| |v(t,x') - v(t,x)| \\ &+ |\delta_w| \left|\frac{\partial g}{\partial y_0}(U^*(t,x')) - \frac{\partial g}{\partial y_0}(U^*(t,x'))\right| |v(t,x)| \\ &\le M_3(1+t^{-\beta}) \|U_0\|_Y |x'-x|^\alpha |\delta_w|, \ 0 \le t \le T, \end{aligned} \tag{8.65}$$

where M_3 is a constant that depends only on T and not on the choice of x or $x' \in \Gamma$. In estimating the first term after the first inequality, we used (8.62). In estimating the second term, we used the fact that $U^*(t) \in \mathcal{A}, t \ge 0$, given that $\mathcal{A}$ is an invariant set. By Lemma 8.6, $\|U^*(t)\|_{(C^\alpha(\Gamma))^{N+1}}$ is finite and is bounded by a constant that does not depend on the choice of solution contained in $\mathcal{A}$. The absolute value $|v(t,x)|$ is bounded by $\|v\|_{C(\Gamma)}$ which in turn is bounded by $M_\Phi \|U_0\|_Y$ by (8.60). The same is thus true for $\frac{\partial g}{\partial y_0}(U^*)$ since $\frac{\partial g}{\partial y_0}$ is a smooth function. I_2 may be estimated as:

$$\begin{aligned} I_2 &\le \delta_w \cdot \frac{\partial g}{\partial y_w}(U^*(t,x'))\delta_w \\ &+ |\delta_w| \left|\frac{\partial g}{\partial y_w}(U^*(t,x')) - \frac{\partial g}{\partial y_w}(U^*(t,x))\right| |w_2(t,x)| \\ &\le -\zeta |\delta_w|^2 + M_4 \|U_0\|_Y |x'-x|^\alpha |\delta_w|. \end{aligned} \tag{8.66}$$

where M_4 is a constant that depends only on T. To estimate the first term after the first inequality we used (8.2). The second term was estimated in the same way as the second term after the first inequality in (8.65). Combining (8.64), (8.65) and (8.66), we have:

$$\frac{1}{2}\frac{\partial}{\partial t}|\delta_w|^2 \le M_5(1+t^{-\beta}) \|U_0\|_Y |x'-x|^\alpha |\delta_w|, \ 0 \le t \le T, \tag{8.67}$$

where M_5 is a constant that depends on T. We have discarded the term $-\zeta\left|\delta_w\right|^2$ in (8.66). Arguing as in the proof of Lemma (7.4) and using the fact that $w_2(0)$ is identically equal to 0, we have:

$$\begin{aligned} |\delta_w(T)| &\leq M_5 \left\|U_0\right\|_Y |x'-x|^\alpha \int_0^T (1+t^{-\beta})dt \\ &\leq M_5 \left(T+\frac{1}{1-\beta}T^{1-\beta}\right) \left\|U_0\right\|_Y |x'-x|^\alpha . \end{aligned} \tag{8.68}$$

This, together with (8.62) shows that $L_2(T)$ is a bounded operator from $Y = (C(\Gamma))^{N+1}$ to $(C^\alpha(\Gamma))^{N+1}, 0<\alpha<1$ and hence a compact operator on Y. □

Note that, in the above proof, we made essential use of the fact that $\mathcal{A}$ is bounded in $(C^\alpha(\Gamma))^{N+1}$ to show that $\Phi_L(U_0^*,T)$, $U_0^*\in\mathcal{A}$ can be decomposed into a contraction and a compact map. It is not clear whether $\Phi_L(U_0^*,T)$ admits such a decomposition if $U_0^*\notin\mathcal{A}$.

In [22], the author considers the asymptotic behavior of the classical system in which $-\Lambda$ in (2.14) is replaced by the standard Laplacian in a bounded domain. The author shows that the global attractor for the FitzHugh-Nagumo and Hodgkin-Huxley systems in this classical setting possess global attractors of finite dimension in an L^2 framework. The author does not discuss the regularity of the global attractor. The methods used in this section can be easily adapted to this classical case, and we can thus conclude that the global attractor is smooth for these classical systems. We also note that the nonlinearities handled here are more general than those in [22]. In [22], the author treats only the case when $g(v,w)$ is a linear function of w.

9. Size Independent Bounds and the Small Cell Limit

In many situations, the conductances σ_i and σ_e of (1.1a) and (1.1b) are "large', or equivalently, the cell is "small". In such cases, one often assumes that the solutions are spatially homogeneous. In this section, we discuss the justification of such assumptions by studying the "small cell limit", the meaning of which will be specified below. The first step in studying this limit is to establish L^∞ bounds on mild solutions that is independent of the cell size $L>0$ for sufficiently small L. We shall in fact obtain a more general result. For HH-type systems and the FitzHugh-Nagumo system, we obtain L^∞ bounds on the solutions that are independent of $0<L<\infty$, not just for small L. As we shall see shortly, such bounds are immediate if $-\Lambda$ satisfies the positivity principle. The interesting point here is that size-independent bounds can be obtained even in the general quasipositive case.

For each $L>0$, consider the boundary value problem (2.1a)-(2.1f) on the domain Ω_i^L and Ω_e^L demarcated by the smooth surface Γ^L:

$$\Gamma^L = \{x\in\mathbb{R}^3 | x/L\in\Gamma,\ L>0\}. \tag{9.1}$$

This is just the L-dilation of the surface Γ. Denote the solutions to this problem by $v^L, w^L = (w_1^L,\cdots,w_N^L)$. Define the operator Λ_σ^L as in (2.5a)-(2.5c) and (2.8) where Ω_i,Ω_e,Γ are replaced respectively by their L-dilations. One may analogously define the L-dilation of the one-phase problem (2.3a)-(2.3c), its solutions v^L, w_k^L and the operator Λ_i^L of (2.11). We refer to $\Lambda_\sigma^L,\Lambda_i^L$ collectively as Λ^L. The functions v^L, w^L clearly satisfy (2.14a) and (2.14b) where Λ in (2.14a) is now Λ^L.

Consider a smooth function $u^L(x)$ defined on Γ^L. Let

$$\tilde{u}(x) = u^L(Lx),\ x \in \Gamma. \tag{9.2}$$

Note that

$$(\Lambda_\sigma^L u^L)(Lx) = \frac{\partial u_i^L}{\partial \mathbf{n}}(Lx) = \frac{1}{L}\frac{\partial \tilde{u}_i}{\partial \mathbf{n}}(x) = \frac{1}{L}(\Lambda_\sigma \tilde{u})(x). \tag{9.3}$$

Here, $\tilde{u}_i$ denote the solutions to the problems (2.5a)-(2.5c) in Ω_i where u is replaced by $\tilde{u}$. u_i^L denotes the solution to the corresponding problem in the L-dilated domain where u is replaced by u^L. A similar calculation can be made for the operator Λ_i^L and Λ_i. Define

$$\tilde{v}(x) = v^L(Lx),\ \tilde{w}_k(x) = w_k^L(Lx),\ x \in \Gamma. \tag{9.4}$$

From (9.3), we see that $\tilde{v}$ and $\tilde{w} = (\tilde{w}_1, \cdots, \tilde{w}_N)$ satisfy

$$\frac{\partial v}{\partial t} = -\frac{1}{L}\Lambda v + f(v,w), \tag{9.5a}$$

$$\frac{\partial w}{\partial t} = g(v,w), \tag{9.5b}$$

where we dropped the $\tilde{\cdot}$ to avoid cluttered notation. The above equations are defined on Γ. We thus see that dilating the domain by a factor of L has the same effect as multiplying the conductances σ_i and σ_e in (1.1a) and (1.1b) by a factor of $1/L$.

9.1. **Size-Independent Bounds.** We prove bounds on mild solutions of (9.5) in that is uniform with respect to L. We first consider the case of HH-type systems.

Theorem 9.1. *Suppose f and g in* (9.5) *are of HH-type. Take a constant $M_0 > 0$ and let let $U_0 \in \mathcal{U}_{M_0} \cap Y$ where Y and $\mathcal{U}_{M_0}$ were defined in* (5.5) *and* (7.7) *respectively. Let $U(t)$ be the mild solution to* (9.5) *with initial data U_0. Then, there is a constant M that depends only on M_0 and is uniform in L such that $U(t), t \geq 0$ lies in $\mathcal{U}_M$.*

Moreover, there is a constant M_*, that is independent of M_0 and $0 < L < \infty$, such that $U(t) \in \mathcal{U}_{M_*}$ for $t > T$ where T depends on M_0 but is independent of $0 < L < \infty$.

Proof. In proving the above, we use both the energy argument of Theorem 7.3 and the invariant rectangle argument of Theorem 6.7.

We first adapt the proof of Theorem 7.3 to (9.5a) and (9.5b). Let $U(t) = (v(t), w(t))$ and assume that the mild solution is smooth. In the general case, we may argue by approximation. Consider:

$$\begin{aligned} &v(t) = I_1 + I_2, \\ &I_1 = \exp(-t\Lambda/L)v(0),\ I_2 = \int_0^t \exp(-(t-s)\Lambda/L)f(v,w)ds. \end{aligned} \tag{9.6}$$

It is easy to see from the derivation of (7.18) that this bound holds for the above $v(t)$ and that the constants C_0 and C_1 do not depend on L.

We also see that the L^2 bound (7.22) holds where the constant C_2 and T_0 are independent of L. Take any constant $T_1 > 0$, and we seek a bound for $\|v(t)\|_{H^s(\Gamma)}, 0 < s < 1$ for $t \geq T_0 + T_1$. We have:

$$\|I_1\|_{H^s(\Gamma)} \leq M_2^{s,0}(1 + L^s T_1^{-s} \|v(t-T_1)\|_{L^2(\Gamma)} \leq M_2^{s,0}(1 + L^s T_1^{-s})C_2. \tag{9.7}$$

Compared to (7.23), we acquire a dependence on L that grows like L^s for large L. Likewise, we obtain:

$$\|I_2\|_{H^s(\Gamma)} \leq C_3(T_1 + L^s T_1^{1-s}) \tag{9.8}$$

similarly to (7.26), where C_3 depends only on C_2. The bounds on I_1 and I_2

$$\|v(t)\|_{H^s(\Gamma)} \leq C_4(1 + L^s) \tag{9.9}$$

where C_4 depends on T_1.

Using this bound, we may proceed to estimate $\|v(t)\|_{W_p^\sigma(\Gamma)}$ similarly to (7.28):

$$\|v(t)\|_{W_p^\sigma(\Gamma)} \leq C_5(1 + L^{s+\sigma}) \tag{9.10}$$

where C_5 depends only on T_1. By the embedding $W_p^\sigma(\Gamma) \hookrightarrow C(\Gamma)$ for $\sigma > 2/p$, we have:

$$\|v(t)\|_{C(\Gamma)} \leq C_6(1 + L^{s+\sigma}),\ t \geq T_0 + 2T_1, \tag{9.11}$$

where C_6 depends only on T_1. Note that $s + \sigma > 1$. This bound is well-behaved for small values of L.

We now derive a bound that is well-behaved for large values of L. For this, we adapt the proof of Theorem 6.7. Take $L_0 > 0$ large enough so that

$$\frac{2\beta(-\Lambda)}{L_0} \leq \frac{\gamma}{2} \tag{9.12}$$

where γ is the constant that appears in (7.5). Suppose $\|v(t_0)\|_{C(\Gamma)} = M_1$ at some time $t = t_0$, and suppose $v(t_0, x_0) = M_1$. Then,

$$\left.\frac{\partial v}{\partial t}\right|_{(t,x)=(t_0,x_0)} = \frac{2\beta(-\Lambda)}{L} M_1 + f(v, w) \leq \frac{2\beta(-\Lambda)}{L} M_1 - \gamma M_1 + \eta \tag{9.13}$$

where we used (7.5). Suppose $L \geq L_0$, and pick a $\delta > 0$.

$$\left.\frac{\partial v}{\partial t}\right|_{(t,x)=(t_0,x_0)} \leq -\frac{\gamma}{2} M_1 + \eta \leq -\delta \text{ if } M_1 \geq 2(\eta + \delta)/\gamma \equiv M_2. \tag{9.14}$$

A similar expression for $\frac{\partial v}{\partial t}$ can be obtained for a point that satisfies $v(t_0, x_0) = -M_1$. Now, suppose $M_0 \leq M_2$. Then, we see immediately that $U(t) \in \mathcal{U}_{M_2}$ for $t \geq 0$. When $M_0 > M_2$, by adopting the same argument as Theorem 6.7, we see that $U(t) \in \mathcal{U}_{M_2}$ for $t \geq (M_0 - M_2)/\delta$. We thus have:

$$\|v(t)\|_{C(\Gamma)} \leq M_2 \text{ for } t \geq (M_0 - M_2)/\delta, L \geq L_0. \tag{9.15}$$

Combining (7.18), (9.11) and (9.15), we obtain the desired result. □

Recall condition (7.4) in Definition 7.2 of HH-type systems. It is possible to prove an analogue of the above theorem for small L even without this global Lipschitz condition, provided that Ω_i is connected. We defer the statement and proof of this result to Appendix B.

The above theorem does not directly apply to FN type systems since such systems do not necessarily possess an invariant cylinder. We have found it difficult to obtain an analogue of the above theorem for FN-type systems. However, as far as the classical FitzHugh-Nagumo system (6.27) is concerned, one can establish an analogue of Theorem 9.1 under the condition that Ω_i is connected.

For $M > 0$, define

$$\begin{aligned} &\mathcal{V}_M = \{U = (v, w) \in (C(\Gamma))^2 | \, \|v\|_{C(\Gamma)} \leq M, \|w\|_{C(\Gamma)} \leq qM\}, \\ &\quad q : \text{constant greater than } \theta/\mu, \end{aligned} \tag{9.16}$$

where θ and μ are the constants that appear in (6.27b). Note that q is the same as the constant that appears in the proof of Corollary 6.8.

Theorem 9.2. *Suppose Ω_i is connected and let f and g be the FitzHugh-Nagumo nonlinearities defined in* (6.27). *Take $M_0 > 0$ and let the initial condition $U_0 \in \mathcal{V}_{M_0} \cap X^2$ and let $U(t)$ be the corresponding the mild solution of* (9.5). *Then there is a constant M depends only on M_0 that is independent of L such that $U(t) \in \mathcal{V}_M$ for $t \geq 0$.*

Unlike Theorem 9.1, the above theorem does not imply an asymptotic uniform bound independent of L. This will be obtained as a direct consequence of Theorem 9.3.

Proof. We first obtain a bound that is uniform for small values of L. Take $M_v > M_0$ to be determined later, and suppose that $\|v(t)\|_{C(\Gamma)} \geq M_v$ for some $t > 0$. Define t_v as:

$$t_v = \inf_{t \in \mathcal{A}} t, \ \mathcal{A} = \{t \in [0, t_0) | \ \|v(t)\|_{C(\Gamma)} \geq M_v\}. \tag{9.17}$$

Given

$$w(t) = w(0) \exp(-\mu t) + \int_0^t \exp(-\mu(t-s))\theta v(s) ds, \tag{9.18}$$

we have

$$\|w(t)\|_{C(\Gamma)} \leq qM_0 \exp(-\mu t) + \theta\mu^{-1} M_v (1 - \exp(-\mu t)) \leq qM_v \tag{9.19}$$

for any $t \in [0, t_v]$. We used $U(0) \in \mathcal{V}_{M_0}$, $\theta/\mu < q$ and $M_v > M_0$. Thus, $U(t) \in \mathcal{V}_{M_v}$ up to $t = t_v$.

Take the variation of constants formula for (9.5a):

$$v(t) = \exp(-\Lambda t/L) v^0 + \int_0^t \exp(-\Lambda(t-s)/L) f(v, w) ds. \tag{9.20}$$

Take the norm in $C(\Gamma)$ on both sides of the above and take $t = t_v$:

$$\begin{aligned} M_v = \|v(t_v)\| &\leq M_c M_0 + \int_0^{t_v} M_c K(M_v) ds = M_c M_0 + t_v M_c K(M_v), \\ K(M_v) &= \sup_{|y_0| < M_v} f_{\mathrm{FN}}(y_0) + qM_v \end{aligned} \tag{9.21}$$

where we used equation (4.74), $f(v, w) = f_{\mathrm{FN}}(v) - w$ and (9.19). Therefore, we have the following estimate for t_v:

$$t_v \geq \frac{M_v - M_c M_0}{M_c K(M_v)}. \tag{9.22}$$

We now proceed to estimate $\mathcal{P}v(t)$ at $t = t_v$ in the $C(\Gamma)$ norm. Apply $\mathcal{P}$ to (9.6).

$$\begin{aligned} &\mathcal{P}v(t) = I_1 + I_2, \\ &I_1 = \exp(-\Lambda t/L)\mathcal{P}v^0, \ I_2 = \int_0^t \exp(-\Lambda(t-s)/L)(\mathcal{P}f(u, w)) ds. \end{aligned} \tag{9.23}$$

For I_1, we have:

$$\|I_1\|_{C(\Gamma)} \leq M_c \exp(-\lambda t/L) \left\|\mathcal{P}v^0\right\|_{C(\Gamma)} \leq 2M_c M_0 \exp(-\lambda t/L). \tag{9.24}$$

where we used (4.81). For I_2, we have:

$$\|I_2\|_{C(\Gamma)} \leq M_c \int_0^t \exp(-\lambda(t-s)/L) \left(\|\mathcal{P}f_{\mathrm{FN}}(v)\|_{C(\Gamma)} + \|\mathcal{P}w\|_{C(\Gamma)}\right) ds. \tag{9.25}$$

Using (9.19), we see that:

$$\|\mathcal{P}w\|_{C(\Gamma)} \leq 2\|w\|_{C(\Gamma)} \leq 2qM_v. \tag{9.26}$$

$\|\mathcal{P}f_{\mathrm{FN}}(v)\|_{C(\Gamma)}$ may be as follows. First, note that:

$$|f_{\mathrm{FN}}(v) - f_{\mathrm{FN}}(\overline{v})| \leq \left|\int_{\overline{v}}^{v} \frac{\partial f_{\mathrm{FN}}}{\partial y_0}(\sigma)d\sigma\right| \leq \left(\sup_{y_0 < M_v}\left|\frac{\partial f}{\partial y_0}\right|\right)|v - \overline{v}|. \tag{9.27}$$

Thus,

$$\begin{aligned}
\|\mathcal{P}f_{\mathrm{FN}}(v)\|_{C(\Gamma)} &= \left\|f_{\mathrm{FN}}(v) - \overline{f_{\mathrm{FN}}(v)}\right\|_{C(\Gamma)} \\
&= \|f_{\mathrm{FN}}(v) - f_{\mathrm{FN}}(\overline{v})\|_{C(\Gamma)} \leq K_f(M_v)\|\mathcal{P}v\|_{C(\Gamma)}, \\
K_f(M_v) &= \sup_{|y_0|<M_v}\left|\frac{\partial f_{\mathrm{FN}}(y_0)}{\partial y_0}\right|.
\end{aligned} \tag{9.28}$$

where we used the fact that $\mathcal{P}$ annihilates constants in the second equality. Substituting (9.26) and (9.28) into (9.25), we obtain a bound for $\|I_2\|_{C(\Gamma)}$. This, combined with (9.24) yields the following:

$$\begin{aligned}
\|\mathcal{P}v\|_{C(\Gamma)} \leq& 2M_c \exp(-\lambda t/L)M_0 + 2M_c q M_v \frac{L}{\lambda} \\
&+ M_c K_f(M_v) \int_0^t \exp(-\lambda(t-s)/L)\|\mathcal{P}v\|_{C(\Gamma)} ds.
\end{aligned} \tag{9.29}$$

Denote the integral in (9.29) by Q. Q satisfies the inequality:

$$\begin{aligned}
\frac{dQ}{dt} &= -\frac{\lambda}{L}Q + \|\mathcal{P}v\|_{C(\Gamma)} \leq -\frac{\lambda_M}{L}Q + R, \\
\lambda_M(L, M_v) &= \lambda - LM_c K_f(M_v), \\
R &= 2M_c \exp(-\lambda t/L)M_0 + 2M_c q M_v \frac{L}{\lambda},
\end{aligned} \tag{9.30}$$

where we used (9.29) in the above inequality. We can solve the differential inequality for Q and substitute this back into (9.29) to obtain the following.

$$\begin{aligned}
\|\mathcal{P}v\| \leq& 2M_c M_0(\exp(-\lambda t/L) + \exp(-\lambda_M t/L)) \\
&+ 2M_c q M_v \frac{L}{\lambda}\left(1 + M_c K_f(M_v)\frac{L}{\lambda_M}\right).
\end{aligned} \tag{9.31}$$

Suppose $v(t_v, z) = M_v$. Using Corollary 4.16 we have:

$$\begin{aligned}
\left.\frac{\partial v}{\partial t}\right|_{(t_v,z)} &\leq 2\beta(-\Lambda)L^{-1}\|\mathcal{P}v\|_{C(\Gamma)} + f_{\mathrm{FN}}(v(t_v,z)) - w(t_v,z) \\
&\leq 2\beta(-\Lambda)L^{-1}\|\mathcal{P}v\|_{C(\Gamma)} - f_{\mathrm{FN}}(M_v) + qM_v.
\end{aligned} \tag{9.32}$$

Using (9.31),

$$\begin{aligned}\left.\frac{\partial v}{\partial t}\right|_{(t_v,z)} &\leq R_0^+(M_v)+R_1(L)+R_2(L),\\ R_0^+(M_v)&=(4\beta(-\Lambda)M_c\lambda^{-1}+1)qM_v+f_{\mathrm{FN}}(M_v),\\ R_1(L)&=4\beta(-\Lambda)qM_vM_c^2K_f(M_v)\frac{L}{\lambda\lambda_M},\\ R_2(L)&=4\beta(-\Lambda)M_cM_0L^{-1}(\exp(-\lambda t_v/L)+\exp(-\lambda_M t_v/L)).\end{aligned} \tag{9.33}$$

Similarly, if $v(t_v,z)=-M_v$, we have:

$$\begin{aligned}\left.\frac{\partial v}{\partial t}\right|_{(t_v,z)} &\geq R_0^-(M_v)-R_1(L)-R_2(L),\\ R_0^-(M_v)&=-(4\beta(-\Lambda)M_c\lambda^{-1}+1)qM_v+f_{\mathrm{FN}}(-M_v),\end{aligned} \tag{9.34}$$

where $R_1(L)$ and $R_2(L)$ are the same as in (9.33). Take a constant $\tilde{M}_v$ so that:

$$\text{If } c>\tilde{M}_v, \text{ then } R_0^+(c)<0 \text{ and } R_0^-(-c)>0. \tag{9.35}$$

Such a constant exists since f_{FN} has cubic growth. Take

$$M_v>\max(M_cM_0,\tilde{M}_v). \tag{9.36}$$

Note that M_v depends only on M_0. Note that $t_v>0$ by (9.22) where the lower bound depends only on M_0. Therefore, there is a $L_c>0$ so that both $R_0^{\pm}(M_v)+R_1(L)+R_2(L)$ are negative and positive respectively for $L\leq L_c$. This shows that $\|v(t_v)\|_{C(\Gamma)}=M_v$ is impossible, a contradiction. By (9.19), we see that $U(t)\in\mathcal{V}_{M_v}, t\geq 0$ for $L\leq L_c$.

For large values of L, we may use Corollary 6.8. Pick a constant $\tilde{M}_v>1$ so that

$$f_{\mathrm{FN}}(\tilde{M}_v)+(2\beta(-\Lambda)/L_c+q)M_v<0. \tag{9.37}$$

Once we have chosen such an $\tilde{M}_v$, the above inequality is satisfied if we replace L_c by any $L>L_c$. Consider the following rectangle in $\mathbb{R}^2$:

$$(y_0,y_1)\in\mathbb{R}^2,\ |y_0|\leq\tilde{M},|y_1|\leq q\tilde{M},\ \tilde{M}\in[\tilde{M}_v,\infty). \tag{9.38}$$

According to Corollary 6.8, the above is a κ-contracting rectangle of the system (9.5a) and (9.5b) if $L\geq L_c$. Thus, if we take:

$$M_v'=\max(\tilde{M}_v,M_0) \tag{9.39}$$

then, $U(t)\in\mathcal{V}_{M_v'}, t\geq 0$ for $L\geq L_c$.

If we let $M=\max(M_v,M_v')$, we obtain the desired result. □

9.2. **Exponential Decay to Spatially Homogeneous Solutions.** In this subsection, we show that the solutions to (9.5) decay exponentially to a spatially homogeneous solution if the cell size L is sufficiently small. We assume that Ω_i is connected and that g satisfies the structure condition (8.2). Let

$$y,y'\in\mathcal{Y}, y=(y_0,y_1,\cdots,y_N)=(y_0,y_w), \tag{9.40}$$

where $\mathcal{Y}$ was defined in (8.1). We adopt a similar coordinate notation for y'.

Theorem 9.3. *Let Ω_i be connected. Consider the system* (9.5)*. Assume either of the following conditions:*

(a) f and g are of HH-type and g satisfies (8.2).

(b) f and g are given by the FitzHugh-Nagumo nonlinearities defined in (6.27).

Take any constant $M_0 > 0$. For conditions (a) and (b) above, let the initial condition satisfy $U_0 \in \mathcal{U}_{M_0}$ and $U_0 \in \mathcal{V}_{M_0}$ respectively, where $\mathcal{U}_{M_0}$ was defined in (7.7) *and $\mathcal{V}_{M_0}$ in* (9.16)*. Denote by $U(t) = (v(t), w(t))$ the corresponding mild solution. Pick positive constants $\xi < \zeta$ and $\nu < \lambda$ where ζ is defined in* (8.2) *and λ is the smallest positive eigenvalue of Λ. Then, there is a constant $L_c > 0$ that depends only on M_0, ξ and ν such that for all positive $L < L_c$, $U(t)$ satisfies the following bounds:*

$$\|\mathcal{P}v(t)\|_{C(\Gamma)} \le K_v(\exp(-\nu t/L) + L\exp(-\xi t)), \tag{9.41}$$

$$\|\mathcal{P}w(t)\|_{(C(\Gamma))^N} \le K_w \exp(-\xi t), \tag{9.42}$$

where K_v and K_w are positive constants that depend only on ξ, ν and M_0. The functions v and w therefore converge to their spatial averages $\overline{v}$ and $\overline{w}$ uniformly at an exponential rate. Moreover, $\overline{v}$ and $\overline{w}$ satisfy the differential equations:

$$\frac{\partial \overline{v}}{\partial t} = f(\overline{v}, \overline{w}) + R_v(t),\ |R_v| \le K_R \exp(-\xi t), \tag{9.43}$$

$$\frac{\partial \overline{w}}{\partial t} = g(\overline{v}, \overline{w}) + R_w(t),\ |R_w| \le K_R \exp(-\xi t). \tag{9.44}$$

The positive constant K_R depends only on ξ and M_0.

Under condition (a), f is required to satisfy the global Lipschitz condition (7.4) as given in Definition 7.2. This is not required for the above conclusion to hold and this refinement is discussed in Appendix B.

The above result may be interpreted as justification for using ODEs in modeling studies of electrophysiology when the cell is small or the conductance is large. Both the membrane potential v and the gating variables w decay exponentially to a spatially homogeneous value, and this value satisfies an ordinary differential equation with exponentially decaying error terms. It is interesting to note that $v - \overline{v}$ quickly decays to a magnitude on the order of L. After this, the decay rate is dictated by the time constants governing the evolution of the gating variables w.

An important ingredient in the proof to follow is the $C(\Gamma)$ bound on the solution that is uniform with respect to the cell size L. We proved such a bound in Theorem 9.1. With this bound, we may assume that the nonlinearities f and g are globally Lipschitz continuous with Lipschitz constant independent of L. This allows us to prove exponential decay of $\mathcal{P}v$ and $\mathcal{P}w$ in an energy norm, which is then combined with parabolic regularity to obtain decay estimates in $C(\Gamma)$.

Proof of Theorem 9.3. We first take U_0 to be smooth, in which case $U(t)$ is smooth. In the sequel we let $\mathcal{W}_M$ denote either $\mathcal{U}_M$ or $\mathcal{V}_M$ depending on whether condition 1 or condition 2 is satisfied.

By Theorem 9.1, or Theorem 9.2, we see that there is a constant M that depends only on M_0 and is uniform in L such that $U(t) \in \mathcal{W}_M, t \ge 0$. Fix such an M.

Consider equations (9.5a) and (9.5b) to which we apply the operator $\mathcal{P}$.

$$\frac{\partial}{\partial t}(\mathcal{P}v) = -\frac{1}{L}\Lambda(\mathcal{P}v) + \mathcal{P}f(v, w), \tag{9.45}$$

$$\frac{\partial}{\partial t}(\mathcal{P}w) = \mathcal{P}g(v, w). \tag{9.46}$$

The operator $\mathcal{P}$ acts componentwise if acting on functions defined on Γ with values in $\mathbb{R}^N, N > 1$. Take the inner product of the above equations with $\mathcal{P}v$ and $\mathcal{P}w$ respectively.

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{P}v\|^2_{L^2(\Gamma)} = -\frac{1}{L}\langle \mathcal{P}v, \Lambda(\mathcal{P}v)\rangle_{L^2(\Gamma)} + \langle \mathcal{P}v, \mathcal{P}f(v,w)\rangle_{L^2(\Gamma)}, \tag{9.47}$$

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{P}w\|^2_{(L^2(\Gamma))^N} = \langle \mathcal{P}v, \mathcal{P}g(v,w)\rangle_{(L^2(\Gamma))^N}. \tag{9.48}$$

We see from the proof of Proposition 3.1 that Λ is a positive definite self-adjoint operator on the image of $\mathcal{P}$ acting on $L^2(\Gamma)$ and $\langle \mathcal{P}v, \Lambda(\mathcal{P}v)\rangle \geq \lambda \|\mathcal{P}v\|^2_{L^2(\Gamma)}$. We now consider the term $\langle \mathcal{P}v, \mathcal{P}f(v,w)\rangle$.

$$\begin{aligned} \langle \mathcal{P}v, \mathcal{P}f(v,w)\rangle_{L^2(\Gamma)} &= \left\langle \mathcal{P}v, f(v,w) - \overline{f(v,w)}\right\rangle_{L^2(\Gamma)} \\ &= \langle \mathcal{P}v, f(v,w) - f(\overline{v},w)\rangle_{L^2(\Gamma)} + \langle \mathcal{P}v, f(\overline{v},w) - f(\overline{v},\overline{w})\rangle_{L^2(\Gamma)}. \end{aligned} \tag{9.49}$$

We used $\left\langle \mathcal{P}v, f(\overline{v},\overline{w}) - \overline{f(v,w)}\right\rangle_{L^2(\Gamma)} = 0$ since $\mathcal{P}v$ is orthogonal to constant functions. Since $U = (v,w)$ are confined to be in $\mathcal{W}_M$, there are Lipschitz constants K_{fv} and K_{fw} that depend only on M (and hence only on M_0) such that

$$\begin{aligned} |f(v,w) - f(\overline{v},w)| &\leq K_{fv}|v - \overline{v}| = K_{fv}|\mathcal{P}v|, \\ |f(\overline{v},w) - f(\overline{v},\overline{w})| &\leq K_{fw}|w - \overline{w}| = K_{fw}|\mathcal{P}w|. \end{aligned} \tag{9.50}$$

This yields:

$$\langle \mathcal{P}v, \mathcal{P}f(v,w)\rangle_{L^2(\Gamma)} \leq K_{fv}\|\mathcal{P}v\|^2_{L^2(\Gamma)} + K_{fw}\|\mathcal{P}v\|_{L^2(\Gamma)}\|\mathcal{P}w\|_{(L^2(\Gamma))^N} \tag{9.51}$$

where we used the Cauchy-Schwarz inequality. Likewise, for $\langle \mathcal{P}w, \mathcal{P}g(v,w)\rangle$ we obtain:

$$\begin{aligned} \langle \mathcal{P}w, \mathcal{P}g(v,w)\rangle_{(L^2(\Gamma))^N} =& \langle \mathcal{P}w, g(v,w) - g(v,\overline{w})\rangle_{(L^2(\Gamma))^N} \\ &+ \langle \mathcal{P}w, g(v,\overline{w}) - g(\overline{v},\overline{w})\rangle_{(L^2(\Gamma))^N}. \end{aligned} \tag{9.52}$$

For the first term on the right hand side, we have:

$$\begin{aligned} \langle \mathcal{P}w, g(v,w) - g(v,\overline{w})\rangle_{(L^2(\Gamma))^N} &= \int_\Gamma (w - \overline{w})\cdot(g(v,w) - g(v,\overline{w}))dS \\ &\leq -\int_\Gamma \zeta|w - \overline{w}|^2 dS = -\zeta\|\mathcal{P}w\|^2_{(L^2(\Gamma))^N} \end{aligned} \tag{9.53}$$

where $\int_\Gamma dS$ denotes integration on the surface Γ, and we used (8.2) in the above inequality. For the second term of (9.52), we have:

$$\langle \mathcal{P}w, g(v,\overline{w}) - g(\overline{v},\overline{w})\rangle_{(L^2(\Gamma))^N} \leq K_{gv}\|\mathcal{P}v\|_{L^2(\Gamma)}\|\mathcal{P}w\|_{(L^2(\Gamma))^N}. \tag{9.54}$$

where K_{gv} is a Lipschitz constant for g with respect to v and hence only depends on M. Substituting these estimates back into (9.47) and (9.48),

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathcal{P}v\|^2_{L^2(\Gamma)} \leq& -(\lambda L^{-1} - K_{fv})\|\mathcal{P}v\|^2_{L^2(\Gamma)}, \\ &+ K_{fw}\|\mathcal{P}v\|_{L^2(\Gamma)}\|\mathcal{P}w\|_{(L^2(\Gamma))^N} \end{aligned} \tag{9.55}$$

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{P}w\|^2_{(L^2(\Gamma))^N} \leq -\zeta\|\mathcal{P}w\|^2_{(L^2(\Gamma))^N} + K_{gv}\|\mathcal{P}v\|_{L^2(\Gamma)}\|\mathcal{P}w\|_{(L^2(\Gamma))^N}. \tag{9.56}$$

Take $p > 2$ small enough so that $\xi_0 \equiv p\xi/2 < \zeta$. Adding the above equations,

$$\begin{aligned}\frac{dE}{dt} \leq & R_E - 2\xi_0 E,\ E = \frac{1}{2}\left(\|\mathcal{P}v\|_{L^2(\Gamma)}^2 + \|\mathcal{P}w\|_{(L^2(\Gamma))^N}^2\right), \\ R_E = & -\left(\lambda L^{-1} - K_{fv}\right)\|\mathcal{P}v\|_{L^2(\Gamma)}^2 - (\zeta - \xi)\|\mathcal{P}w\|_{(L^2(\Gamma))^N}^2 \\ & + (K_{fw} + K_{gv})\|\mathcal{P}v\|_{L^2(\Gamma)}\|\mathcal{P}w\|_{(L^2(\Gamma))^N}.\end{aligned} \tag{9.57}$$

If we take L small enough so that

$$4(\lambda L^{-1} - K_{fv})(\zeta - \xi_0) - (K_{fw} + K_{gv})^2 \geq 0, \tag{9.58}$$

then, $R_E \leq 0$. Thus, for small enough L, we have,

$$\|\mathcal{P}v\|_{L^2(\Gamma)} \leq C_2 \exp(-\xi_0 t),\ \|\mathcal{P}w\|_{(L^2(\Gamma))^N} \leq C_2 \exp(-\xi_0 t) \tag{9.59}$$

where C_2 is a constant that depend only on ξ and M_0. We know that $U = (v, w) \in \mathcal{W}_M$ and thus, $\|\mathcal{P}v\|_{C(\Gamma)}$ and $\|\mathcal{P}w\|_{(C(\Gamma))^N}$ are bounded by constants that depend only on M. By interpolation, we have:

$$\|\mathcal{P}v\|_{L^p(\Gamma)} \leq C_p \exp(-\xi t),\ \|\mathcal{P}w\|_{L^p(\Gamma)^N} \leq C_p \exp(-\xi t) \tag{9.60}$$

where the constant C_p depends only on ξ and M_0. We now use the variation of constants formula as in (9.23) to estimate $\|\mathcal{P}v\|_{C(\Gamma)}$. The term I_1 in (9.23) is estimated similarly to (9.24). The term I_2 in (9.23) is estimated as follows:

$$\|I_2\|_{W_p^s(\Gamma)} \leq \int_0^t M_p^s(1 + (t-\tau)^{-s})\exp(-\lambda(t-\tau)/L)\|\mathcal{P}f(v,w)\|_{L^p(\Gamma)}\,d\tau. \tag{9.61}$$

where we used (4.80). The index s is chosen to satisfy $2/p < s < 1$ so that $W_p^s(\Gamma) \hookrightarrow C(\Gamma)$. From (9.50), it is easily seen that:

$$\|\mathcal{P}f(v,w)\|_{L^p(\Gamma)} \leq K_{fv}\|\mathcal{P}v\|_{L^p(\Gamma)} + K_{fw}\|\mathcal{P}w\|_{(L^p(\Gamma))^N}. \tag{9.62}$$

Using (9.60), we have,

$$\|I_2\|_{W_p^s(\Gamma)} \leq \int_0^t \tilde{C}_p^s\left(1 + (t-\tau)^{-s}\right)\exp(-\lambda(t-\tau)/L)\exp(-\xi t)d\tau, \tag{9.63}$$

where $\tilde{C}_p^s$ is a constant that depends only on ξ and M_0. We have:

$$\int_0^t \exp(-\lambda(t-\tau)/L)\exp(-\xi t)d\tau \leq \frac{L}{\lambda}\exp(-\xi t). \tag{9.64}$$

For $t > L$,

$$\begin{aligned}&\int_0^t \frac{1}{(t-\tau)^s}\exp(-\lambda(t-\tau)/L)\exp(-\xi t)d\tau \\ \leq & \int_{t-L}^t \frac{1}{(t-\tau)^s}\exp(-\xi t)d\tau + L^{-s}\int_0^{t-L}\exp(-\lambda(t-\tau)/L)\exp(-\xi t)d\tau \\ \leq & \left(\frac{1}{1-s} + \frac{1}{\lambda}\right)L^{1-s}\exp(-\xi t).\end{aligned} \tag{9.65}$$

It is clear that this inequality remains valid even if $t \leq L$. This yields an exponentially decaying bound on $\|I_2\|_{W_p^s(\Gamma)}$, and by Sobolev embedding, a similar bound on $\|I_2\|_{C(\Gamma)}$. Combining these bounds with (9.24), we obtain:

$$\|\mathcal{P}v\|_{C(\Gamma)} \leq \tilde{K}_v(\exp(-\lambda t/L) + L^{1-s}\exp(-\xi t)) \leq K_v^0 \exp(-\xi t) \tag{9.66}$$

where $\tilde{K}_v, K_v^0$ depends only on M_0 and ξ. Since $1 - s > 0$, we may take L small enough to make the last inequality valid.

We turn to estimating $\|\mathcal{P}w\|_{C(\Gamma)}$. The following calculations are similar to those in the proof of Lemma 8.3. Take any two points x and x' on Γ and consider the values $(v(t,x), w(t,x))$ and $(v(t,x'), w(t,x'))$. We shall often omit the dependence on t to avoid cumbersome notation. The function $w(x)$ satisfies:

$$\frac{dw}{dt}(x) = g(v(x), w(x)) \tag{9.67}$$

and likewise for $w(x')$. Therefore,

$$\frac{d}{dt}(w(x) - w(x')) = g(v(x), w(x)) - g(v(x'), w(x')). \tag{9.68}$$

Taking the inner product on both sides with respect to $w(x) - w(x')$, we have:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left|w(x) - w(x')\right|^2 = &(w(x) - w(x')) \cdot (g(v(x), w(x)) - g(v(x), w(x'))) \\ &+ (w(x) - w(x')) \cdot (g(v(x), w(x')) - g(v(x'), w(x'))).\end{aligned} \tag{9.69}$$

The first term can be estimated using (8.2). To estimate the second term, note that

$$|v(x) - v(x')| \leq \sup_{x_1, x_2 \in \Gamma} |v(x_1) - v(x_2)| \leq 2\left\|\mathcal{P}v\right\|_{C(\Gamma)} \leq 2K_v^0 \exp(-\xi t). \tag{9.70}$$

where we used (9.66) in the last inequality. The Lipschitz continuity of g allows us to estimate the second term using the above bound on $|v(x) - v(x')|$. We have:

$$\frac{1}{2}\frac{d}{dt}\left|w(x) - w(x')\right|^2 \leq -\zeta\left|w(x) - w(x')\right|^2 + 2K_{gv}K_v^0 \exp(-\xi t)\left|w(x) - w(x')\right|. \tag{9.71}$$

Proceeding as in the proof of Lemma 7.4, we obtain:

$$|w(x) - w(x')| \leq |w(0,x) - w(0,x')| \exp(-\zeta t) + \frac{2K_{gv}K_v^0}{\zeta - \xi}(\exp(-\xi t) - \exp(-\zeta t)). \tag{9.72}$$

Since $\xi < \zeta$, there is a constant K_w that depends only on M_0 and ξ such that

$$|w(x) - w(x')| \leq K_w \exp(-\xi t). \tag{9.73}$$

Since $x, x' \in \Gamma$ were arbitrary,

$$\|\mathcal{P}w\|_{(C(\Gamma))^N} \leq \sup_{x, x' \in \Gamma} |w(x) - w(x')| \leq K_w \exp(-\xi t). \tag{9.74}$$

This is (9.42).

We now use our newly found bound on $\|\mathcal{P}w\|_{(C(\Gamma))^N}$ to obtain (9.41). We follow an argument used in the proof of Theorem 9.2. Using exactly the same calculations leading up to (9.29) and using (9.42),

$$\begin{aligned}\|\mathcal{P}v\|_{C(\Gamma)} \leq &2M_c \exp(-\lambda t/L) M_0 + M_c K_{fw} K_w \frac{L}{\lambda} \exp(-\xi t) \\ &+ M_c K_{fv} \int_0^t \exp(-\lambda(t-s)/L) \left\|\mathcal{P}v\right\|_{C(\Gamma)} ds.\end{aligned} \tag{9.75}$$

This integral inequality may be solved for $\|\mathcal{P}v\|_{C(\Gamma)}$. Proceeding as in the calculations that lead to (9.31), we obtain (9.41).

When U_0 is not smooth, inequalities (9.41) and (9.42) can be established by an approximation argument.

We now turn to (9.43) and (9.44). First, note that, for any function $u \in C(\Gamma)$:

$$\overline{\exp(-t\Lambda)u} = \overline{u}. \tag{9.76}$$

This is most easily seen in the L^2 setting, since 0 is an eigenvalue of Λ and the map $u \mapsto \overline{u}$ is the corresponding spectral projection (see also Proposition 4.14). Take the spatial average in the variation of constants formula for v.

$$\begin{aligned}\overline{v(t)} &= \overline{\exp(-t\Lambda/L)v(0)} + \overline{\int_0^t \exp(-(t-s)\Lambda/L)f(u(s),v(s))ds} \\ &= \overline{v(0)} + \int_0^t \overline{f(u(s),v(s))}ds\end{aligned} \tag{9.77}$$

Given that $U = (u,v) \in C([0,\infty),(C(\Gamma))^{N+1})$, $\overline{f(u(t),v(t))}$ is a continuous function of t. We see that $\overline{v(t)}$ is continuously differentiable, and we may thus take the derivative of the above with respect to t:

$$\begin{aligned}\frac{d\overline{v}}{dt} &= \overline{f(u,v)} = f(\overline{u},\overline{v}) + R_v, \\ R_v &= \frac{1}{|\Gamma|}\int_\Gamma (f(u,v) - f(\overline{u},\overline{v}))dS_x\end{aligned} \tag{9.78}$$

The term R_v can be estimated using the Lipschitz continuity of f as well as the exponential estimates (9.42) and (9.66). A similar calculation can be performed to obtain (9.44). □

We have the following immediate corollary.

Corollary 9.4. *Suppose the hypotheses of 9.3 are satisfied. Consider the semiflow $\Phi_L(t)$ defined by the mild solutions of* (9.5a) *and* (9.5b) *in the space of continuous functions (see Section 8.2). There is a constant $L_c > 0$ such that, for $L < L_c$, the global attractor of the semiflow $\Phi_L(t)$ coincides with the set of all bounded entire solutions of the ODE system:*

$$\begin{aligned}\frac{dv}{dt} &= f(u,v), \\ \frac{dw}{dt} &= g(u,v).\end{aligned} \tag{9.79}$$

Proof. Take L_c small enough so that the conclusions of Theorem 9.3 hold. Given the estimate (9.41) and (9.42) and the fact that the constants K_v and K_w depend only on M_0, we find, similarly to the proof of Lemma 8.6, that the ω-limit set of any bounded set must consist only of constant functions. Constant functions that are mild solutions to (9.5a) and (9.5b) satisfy (9.79) (see (9.78)). The global attractor consists of all bounded entire solutions to (9.5a) and (9.5b) and hence of (9.79). □

Appendix A. Positivity of $-\Lambda_\sigma$ when Γ is an Ellipsoid

We prove the following proposition.

Proposition A.1. *Let Γ be an ellipsoid whose axes are given by $a_1 \geq a_2 \geq a_3$. The operator $-\Lambda_1$ satisfies the positivity principle if and only if*

$$\frac{a_1}{a_3} \leq \sqrt{2} + \sqrt{3}. \tag{A.1}$$

Proof. As we saw in the discussion immediately after the proof of Proposition 4.8, we must find the condition under which (4.36) is non-positive for all choices of $x, y \in \Gamma$. Let A be:

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \tag{A.2}$$

Then, (4.36) may be written as:

$$I_{xy} = -\frac{A^{-2}x}{|A^{-2}x|} \cdot \frac{A^{-2}y}{|A^{-2}y|} + 3\left(\frac{A^{-2}x}{|A^{-2}x|} \cdot \frac{x-y}{|x-y|}\right)\left(\frac{A^{-2}y}{|A^{-2}x|} \cdot \frac{x-y}{|x-y|}\right) \tag{A.3}$$

where x and y must satisfy the constraints:

$$\left|A^{-1}x\right| = \left|A^{-1}y\right| = 1. \tag{A.4}$$

Non-positivity of the above is equivalent to the non-negativity of:

$$\begin{aligned} f(x,y) &= (A^{-2}x \cdot A^{-2}y)\,|x-y|^2 - 3(A^{-2}x \cdot (x-y))(A^{-2}y \cdot (x-y)) \\ &= (A^{-2}x \cdot A^{-2}y)\,|x-y|^2 + \frac{3}{4}\left|A^{-1}(x-y)\right|^4 \end{aligned} \tag{A.5}$$

where we used the (A.4) in the second equality. We thus seek the minimum of the above quantity under the constraints (A.4). Consider the function:

$$g(x,y) = f(x,y) - \lambda(\left|A^{-1}x\right|^2 - 1) - \mu(\left|A^{-1}y\right|^2 - 1) \tag{A.6}$$

where λ and μ are Lagrange multipliers. At the critical points, the gradients of g with respect to x and y, $\nabla_x g$ and $\nabla_y g$ respectively, are equal to 0. A routine computation shows that:

$$(x+y)\cdot(\nabla_x g + \nabla_y g) = \left|A^{-2}(x+y)\right|^2 |x-y|^2 - 2(\lambda+\mu)\left|A^{-1}(x+y)\right|^2 = 0. \tag{A.7}$$

On the other hand, we have:

$$x \cdot \nabla_x g + y \cdot \nabla_y g = x \cdot \nabla_x f + y \cdot \nabla_y f - 2\lambda - 2\mu = 4f - 2(\lambda+\mu) = 0. \tag{A.8}$$

The last equality can be inferred by noting that $f(x,y)$ is a homogeneous 4th order polynomial in x and y. Substituting (A.8) into (A.7), we have:

$$\left|A^{-2}(x+y)\right|^2 |x-y|^2 = 4f\left|A^{-1}(x+y)\right|^2 \tag{A.9}$$

at critical points. Therefore, if $x+y \neq 0$ at critical points, f is always non-negative. We may therefore restrict our attention to critical points such that $x+y=0$. Now,

$$f(x,-x) = -4\left|A^{-2}x\right|^2 |x|^2 + 12. \tag{A.10}$$

We seek the minimum of this quantity under $\left|A^{-1}x\right| = 1$. Let ν be our Lagrange multiplier, and consider the gradient of the function:

$$h(x) = -4\left|A^{-2}x\right|^2 |x|^2 + 12 - \nu(\left|A^{-1}x\right|^2 - 1). \tag{A.11}$$

We have:

$$\nabla_x h = -8|x|^2 A^{-4}x - 8\left|A^{-2}x\right|^2 x - 2\nu A^{-2}x = 0. \tag{A.12}$$

We may simplify the above relation to find:

$$\frac{A^{-4}x}{\left|A^{-2}x\right|^2} + \frac{x}{|x|^2} = 2A^{-2}x. \tag{A.13}$$

Suppose $a_1 > a_2 > a_3$. Let $x = (x_1, x_2, x_3)$ and suppose $x_i \neq 0$ for all $i = 1, 2, 3$. From (A.13) we have:

$$a_i^{-4} z_1 + z_2 = 2a_i^{-2},\ z_1 = \left|A^{-2}x\right|^{-2}, z_2 = |x|^{-2}\ i = 1, 2, 3. \tag{A.14}$$

The above is a set of three linear equation in two unknowns z_1, z_2, and it is easily seen that this has no solution if $a_i, i = 1, 2, 3$ are all different. Therefore, if $a_1 > a_2 > a_3$, then at least one x_i must be zero.

Let us assume that $x_3 = 0$. The constraint $\left|A^{-1}x\right| = 1$ can now be parametrized so that $x_1 = a_1 \cos\theta, x_2 = a_2 \sin\theta$. Substitute this into $f(x, -x)$ and to find the minimum m_3 under the constraint $x_3 = 0$. The value m_3 attained at $x = (\pm a_1/\sqrt{2}, \pm a_2/\sqrt{2}, 0)$ and is equal to:

$$m_3 = -\left(\frac{1}{a_1^2} + \frac{1}{a_2^2}\right)\left(a_1^2 + a_2^2\right) + 10. \tag{A.15}$$

Solving the condition $m_3 \geq 0$, we find that $a_1/a_2 \geq \sqrt{2} + \sqrt{3}$. We can obtain the same conclusions when $x_1 = 0$ or $x_2 = 0$, and we find:

$$\frac{a_i}{a_j} \geq \sqrt{2} + \sqrt{3} \text{ for } i > j. \tag{A.16}$$

This is equivalent to (A.1) given $a_1 > a_2 > a_3$.

If $a_1 = a_2$, we may assume, by symmetry, that $x_1 = 0$ or $x_2 = 0$, and likewise when $a_2 = a_3$. The rest of the computation is the same as above. □

Appendix B. Some Extension of Results in Section 9

We prove an analogue of Theorem 9.1 when f is not necessarily globally Lipschitz.

Proposition B.1. *Suppose Ω_i is connected. Let f satisfy (7.5) and g satisfy the invariant cylinder condition (7.2). Take $M_0 > 0$ and let $U_0 \in \mathcal{U}_{M_0} \cap Y$, where $\mathcal{U}_{M_0}$ is defined as in (7.7). Take any constant $M > M_{crit}$, where M_{crit} is a constant that depends only on M_0. There is a positive constant L_M that depends only on M and M_0 such that for all $L < L_M$ a mild solution $U(t) \in C([0, T]; Y), T > 0$ to system (9.5) exists for all time, and $U(t) \in \mathcal{U}_M$ for $0 \leq t < \infty$.*

Proof. We use an argument similar to that of Theorem 9.2. We first assume that U_0 is smooth. Take a $M > M_0$ to be determined later. Define t_M as:

$$t_M = \inf_{t \in \mathcal{A}} t,\ \mathcal{A} = \{t \in [0, \infty) |\, U(t) \notin \mathcal{U}_M\}. \tag{B.1}$$

Since $U(t, x)$ can never leave the invariant cylinder $\mathcal{C}$, we have only to consider the case when $|v(t, x)|$ reaches M at time t_M.

We first show that if M is taken large enough, $t_M > t_c > 0$ where t_c depends only on M, M_0 and does not depend on L. Performing a calculation similar to the calculation that led to (9.22), we have:

$$t_M \geq \frac{M - M_c M_0}{M_c K(M)} \equiv t_c,\ K(M) = \sup_{y \in \mathcal{C}_M} |f(y)|\,. \tag{B.2}$$

where M_c is the constant that appears in (4.74). If we take $M > M_c M_0$, t_c is positive.

Let $t \leq t_M$. Note that (7.21) is valid in our case, and thus,

$$\|v(t)\|_{L^2(\Gamma)} \leq |\Gamma|^{1/2} \max(M_0, \eta/\gamma). \tag{B.3}$$

We have, therefore:

$$\left|\overline{v(t)}\right| \leq \max(M_0, \eta/\gamma) \equiv C_0.. \tag{B.4}$$

We shall now obtain a bound for $\mathcal{P}v$ using (9.23). Analogously to (9.24), we have:

$$\|I_1\|_{C(\Gamma)} \leq 2M_c \exp(-\lambda t/L) M_0. \tag{B.5}$$

We now turn to I_2.

$$\|I_2\|_{C(\Gamma)} \leq M_c \int_0^t \exp(-\lambda(t-s)/L) \|\mathcal{P}f(u,w)\|_{C(\Gamma)}\, ds \tag{B.6}$$

where we used (4.81). We split $\mathcal{P}(f(v,w))$ into J_1 and J_2 as follows:

$$\begin{aligned} \mathcal{P}f(v,w) &= \mathcal{P}(f(v,w) - f(\overline{v},\overline{w})) = J_1 + J_2, \\ J_1 &= \mathcal{P}(f(v,w) - f(\overline{v},w)),\ J_2 = \mathcal{P}(f(\overline{v},w) - f(\overline{v},\overline{w})), \end{aligned} \tag{B.7}$$

We estimate these in the $C(\Gamma)$ norm.

$$\|J_1\|_{C(\Gamma)} \leq K_f(M) \|\mathcal{P}v\|_{C(\Gamma)},\ K_f(M) = \left(\sup_{y\in\mathcal{C}_M} \left|\frac{\partial f}{\partial y_0}\right|\right). \tag{B.8}$$

For J_2, we have,

$$\|J_2\|_{C(\Gamma)} \leq 2\|f(\overline{v},w) - f(\overline{v},\overline{w})\|_{C(\Gamma)} \leq C_1 \tag{B.9}$$

where C_1 is a constant that depends only on M_0. This follows from the fact that w is bounded by a universal constant ($U(t)$ resides in the invariant cylinder) and the bound (B.4). We thus have the following integral inequality for $\|\mathcal{P}v\|_{C(\Gamma)}$.

$$\begin{aligned} \|\mathcal{P}v\|_{C(\Gamma)} \leq& 2M_c \exp(-\lambda t/L) M_0 + M_c C_1 \frac{L}{\lambda} \\ &+ M_c K_f(M) \int_0^t \exp(-\lambda(t-s)/L) \|\mathcal{P}v\|_{C(\Gamma)}\, ds. \end{aligned} \tag{B.10}$$

We now take any M that satisfies:

$$M > \max(M_c M_0, 2\beta(-\Lambda) M_c C_1 (\lambda\gamma)^{-1}) \equiv M_{\text{crit}} \tag{B.11}$$

where γ is growth constant in (7.5). Solving this differential inequality in the same way as in Theorem 9.2, we have:

$$\begin{aligned} \|\mathcal{P}v\| \leq& 2M_c M_0(\exp(-\lambda t/L) + \exp(-\lambda_M t/L)) \\ &+ M_c C_1 \frac{L}{\lambda} + M_c^2 C_1 K_f(M) \frac{L^2}{\lambda\lambda_M}, \\ \lambda_M(L,M) =& \lambda - L M_c K_f(M). \end{aligned} \tag{B.12}$$

Suppose $v(t_M, z) = M$. Using (9.5a) and proceeding as in Theorem 9.2, we obtain the following:

$$\left.\frac{\partial v}{\partial t}\right|_{(t_M,z)} \leq 2\beta(-\Lambda) L^{-1} \|\mathcal{P}v\|_{C(\Gamma)} + \sup_{y\in\mathcal{C}, y_0 = M} f(y). \tag{B.13}$$

By (7.5),

$$\sup_{y\in y\in\mathcal{C}, y_0=M} f(y) M \leq -\gamma M^2. \tag{B.14}$$

Substituting this and (B.12) into (B.13), we have:

$$\begin{aligned}\left.\frac{\partial v}{\partial t}\right|_{(t_M,z)} &\leq R_0 + R_1(L) + R_2(L),\\ R_0 &= 2\beta(-\Lambda)M_c C_1 \lambda^{-1} - \gamma M,\\ R_1(L) &= 2\beta(-\Lambda)M_c^2 C_1 K_f(M)\frac{L}{\lambda\lambda_M},\\ R_2(L) &= 4\beta(-\Lambda)M_c M_0 L^{-1}(\exp(-\lambda t_M/L) + \exp(-\lambda_M t_M/L)).\end{aligned} \tag{B.15}$$

As we make L small, $R_1(L)$ can be made arbitrarily small. By (B.11), $M > M_c M_0$ and thus by (B.2), $t_M > t_c > 0$. Therefore, $R_2(L)$ can also be arbitrarily small by making L small. By (B.11), R_0 is negative, and therefore, L can be made small enough so that $\left.\frac{\partial v}{\partial t}\right|_{(t_M,z)} < 0$. This is a contradiction. The same argument can be made when $v(t_M, z) = -M$. When the initial condition U_0 is not smooth, we may argue by approximation. □

We have the following corollary.

Corollary B.2. *The conclusions of Theorem 9.3 hold under the assumptions of Proposition B.1.*

Proof. Proposition B.1 provides bounds that are uniform with respect to L for small L. The rest of the proof is identical to that of Theorem 9.3. □

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