1 Saddle-Node Bifurcation

Suppose we have a differential equation with a certain parameter $r$:

$$\frac{dx}{dt} = f(x, r), \ x \in \mathbb{R}^N, r \in \mathbb{R}. \quad (1)$$

We say that we have a bifurcation at $r = r_\ast$ if the behavior of the dynamical system changes qualitatively at $r = r_\ast$. Rather than discuss generalities, it is easier to take a look at examples. Consider the dynamical system:

$$\frac{dx}{dt} = r - x^2 = f(x). \quad (2)$$

When given a differential equation, the first thing to do is to look for steady states. The steady states $x = x_\ast$ satisfy:

$$r - x_\ast^2 = 0 \quad (3)$$

So,

$$x_\ast = \pm \sqrt{r}, \ r \geq 0. \quad (4)$$

Note here that $r \geq 0$ so that there is a solution. Now, let us examine the stability of each of the fixed points. We must compute the derivative of the $f$.

$$\left. \frac{df}{dx} \right|_{x=x_\ast} = -2x_\ast. \quad (5)$$
Figure 1: Left figure: The intersection of the parabolas $y = r - x^2$ (blue curves) with the $x$ axis (red line) are the fixed points of the differential equations (2). For $r > 0$ there are two fixed points whereas for $r < 0$ there are no fixed points. Right figure: bifurcation diagram, which plots the position of the fixed points as a function of $r$.

We thus see the following (see Figure 1):

- $r < 0$: no fixed points,
- $r = 0$: $x^* = 0$, $f'(0) = 0$,
- $r > 0$: $x^* = -\sqrt{r}$, $f'(-\sqrt{r}) = 2\sqrt{r} > 0$, unstable fixed point, $x^* = \sqrt{r}$, $f'(-\sqrt{r}) = -2\sqrt{r} < 0$, stable fixed point.

One way to visualize the bifurcation is to draw the bifurcation diagram (Figure 1, right). Here, we plot the fixed points as a function of $r$. At $r = 0$, two fixed points emerge at the locations $x^* = \pm\sqrt{r}$.

At $r = 0$, we have a bifurcation. The behavior of the system changes qualitatively at $r = 0$. For $r < 0$, we have no fixed point. At $r = 0$, we have one fixed point, at which $f'(x^*) = 0$. For $r > 0$, we have two fixed points, one of which is stable, and the other of which is unstable. This is called a saddle-node bifurcation or a fold bifurcation. The name saddle-node comes from the fact that, at the bifurcation point, a saddle and node come together and annihilate. In one-dimensional systems, there are no saddles, so with the above example, it may not be clear why it should be called a saddle-node bifurcation. The analogue of the above in higher dimension, in fact, always involves a saddle and a node. We shall see examples of this later.

There are different kinds of bifurcations, some of which we will introduce in these notes. One important observation is that, at the parametric value
\( r = r_\ast \) at which the bifurcation takes place, at the fixed point, we have:

\[
 f'(x_\ast) = 0. \tag{7}
\]

This is a general feature of bifurcations. We shall see later how this condition generalizes in higher dimensional systems.

Let us look at another example. Consider:

\[
 \frac{dx}{dt} = r - x - e^{-x} = f(x). \tag{8}
\]

We want to find the fixed points of the above. Let the fixed points be \( x_\ast \). We must solve the equation:

\[
 r - x_\ast - e^{-x_\ast} = 0. \tag{9}
\]

Solving this equation is equivalent to finding the intersection of the graphs:

\[
 y = r - x \quad \text{and} \quad y = e^{-x}. \tag{10}
\]

As it turns out, when \( r < 1 \), the two graphs do not have any common point, and the line \( y = r - x \) lies below \( y = e^{-x} \). When \( r = 1 \), \( y = r - x \) touches \( y = e^{-x} \) at \( x_\ast = 0 \). For \( r > 0 \), the two graphs have two intersection points, \( x_+ \) and \( x_- \) where \( x_+ > 0 \) and \( x_- < 0 \). Let us examine the stability of these fixed points. The derivative of \( f \) is given by:

\[
 f'(x_\ast) = -1 + e^{-x}. \tag{11}
\]

Therefore, we have:

\[
 r < 1 : \quad \text{no fixed points},
\]
\[
 r = 1 : \quad x_\ast = 0, \quad f'(0) = 0,
\]
\[
 r > 1 : \quad x_\ast = x_-, \quad f'(x_-) = -1 + \exp(-x_-) > 0, \quad \text{unstable fixed point},
\]
\[
 x_\ast = x_+, \quad f'(x_+) = -1 + \exp(-x_+) < 0, \quad \text{stable fixed point}. \tag{12}
\]

To determine the signs of the derivative at \( x_\pm \), we used the fact that \( x_+ > 0 \) and \( x_- < 0 \). We see here that at \( r = 1 \), we have a saddle-node bifurcation.

Let us look at an example that is a little more interesting. We consider the following model of population growth of a spruce budworm (adopted from Strogatz, *Nonlinear Dynamics and Chaos*). These worms live in spruce
forests in eastern Canada, and occasional outbreaks of this worm decimates the spruce forests.

Let $x$ be the population of the spruce budworm. We consider the model:

$$\frac{dx}{dt} = r x \left(1 - \frac{x}{K}\right) - \frac{A x^2}{B + x^2}, \quad r, K, A, B > 0. \quad (13)$$

There are two terms to the equation. The first term represents the logistic growth model that we have already seen. The parameter $r$ is the growth rate and $K$ is the carrying capacity. The second term is new, and models predation by birds. Note first that predation increases with $x$: the more insects there are, the greater the number of insects eaten by the birds. The other important feature is that, as $x$ increases, predation by birds hits a ceiling value of $A$. Even when $x$ is very large, predation by birds will never increase above $A$.

Let us examine the fixed points of this equation. Let $x = x_*$ be a fixed point. We must solve the equation:

$$r x_* \left(1 - \frac{x_*}{K}\right) = \frac{A x_*^2}{B + x_*^2}. \quad (14)$$

This means that, we have:

$$x_* = 0 \text{ or } r \left(1 - \frac{x_*}{K}\right) = \frac{A x_*}{B + x_*^2}. \quad (15)$$

We must therefore examine the latter equation. This amounts to finding the intersection points of the graphs:

$$y = r \left(1 - \frac{x}{K}\right) \quad \text{and} \quad y = \frac{A x}{B + x^2}. \quad (16)$$

We consider the specific situation when

$$\frac{A}{2 \sqrt{B}} < r < \frac{3 \sqrt{3} A}{8 \sqrt{B}}. \quad (17)$$

Then, we will have the following picture as $K$ is increased (See Figure 2). For values of small $K$, the two graphs have only one intersection point, $x_-$. When $K$ reaches $K = K_*$, the graphs are tangent to each other at a point $\hat{x}$. When $K$ is increased further, two intersection points, $x_0$ and $x_+$ emerge.
Figure 2: The intersection of the two graphs, $Ax/(B + x^2)$ (red curve) and $r(1-x/K)$ (blue lines) for various values of $K$. As $K$ increases ($x$-intercept of the blue lines) the number of intersection points goes from 3, increases to 0 and decreases back to 1.

At $K = K^*$, the graphs are tangent again, where the points $x_-$ and $x_0$ coalesce to form the tangent point $\tilde{x}$. Above $K = K^*$, we again only have one intersection point $x = x_+$. We can now see how the system behaves for different values of $K$. The stability of each of the fixed points can be determined by taking the derivative, but in this case, it is probably easier to just consider the sign of the right hand side of (13). We have:

$0 < K < K_* : x_+ = 0$ unstable, $x_- stable,$

$K = K_* : x_* = 0, \tilde{x} unstable, x_- stable,$

$K_* < K < K^* : x_+ = 0, x_0 unstable, x_- , x_+ stable,$

$K = K^* : x_* = 0, \tilde{x} unstable, x_+ stable,$

$K^* < K : x_+ = 0 unstable, x_+ stable.$

In this case, we have a saddle-node bifurcation, both at $K = K_*$ and $K = K^*$. We note that condition (17) ensures that we will indeed have two saddle-node bifurcations.

Let us consider the biological implications of this analysis. The parameter $K$ is the carrying capacity of the budworm living in the spruce forest. If
the forest is young, $K$ is low, and $K$ becomes larger as the forest matures. When $K$ is small, there is only one stable steady state, $x_-$. As the forest matures, the value of $K$ crosses $K^*$. Here, $x_-$ disappears in a saddle-node bifurcation, and the only stable steady state is $x_+ = x_+$. As $K$ crosses $K^*$, therefore, we see a sudden increase in the budworm population. This model thus explains the budworm outbreak.