1 Single Variable Optimization

1.1 A sample problem

We first consider the following problem (from p. 52 of Meerschaert).

A manufacturer of personal computers currently sells 10,000 units per month of a basic model. The cost of manufacture is $700/unit and the wholesale price is $950. During the last quarter the manufacturer lowered the price by $100 in a few test markets and the result was a 50% increase in sales. The company has been advertising its product nationwide at a cost of $50,000 per month. The advertising agency claims that increasing the ad budget by $10,000/month would result in a sales increase of 200 units/month. Management has agreed to consider an increase in ad budget to no more than $100,000/month.

Let us introduce some variables.

\[
\begin{align*}
N_0 & \quad \text{current number of units sold per month} \quad \$10,000 \\
p_0 & \quad \text{current price/unit} \quad \$950 \\
c & \quad \text{cost/unit} \quad \$700 \\
q & \quad \text{price lowered in test market} \quad \$100 \\
r & \quad \text{factor of sales increase in test market} \quad 0.5 \\
A_0 & \quad \text{current ad budget per month} \quad \$50,000 \\
a & \quad \text{ad budget increment per month} \quad \$10,000 \\
b & \quad \text{increase in sales per ad budget increment} \quad 200 \text{ units} \\
A_1 & \quad \text{maximum ad budget} \quad \$100,000 \\
\end{align*}
\]

Let \( p \) be the price of the unit. We have:

\[ p = p_0 - qD \]  \hspace{1cm} (1)

where \( D \) is the discount rate relative to \( q \). Let \( N \) be the number of units sold. According to the problem, we have:

\[ N = N_0(1 + rD) + bB \]  \hspace{1cm} (2)

where \( B \) is the rate of increase in ad budget relative to \( a \). Note that we have assumed that discounting leads to a linear increase in sales (this is an assumption that, strictly speaking, is not in the problem statement). The total cost \( C \) per month is given by:

\[ C = cN + A_0 + aB. \]  \hspace{1cm} (3)
The revenue $R$ is given by:

$$R = pN. \quad (4)$$

The profit $P$ is given by:

$$P = R - C = (p - c)N - A_0 - aB$$

$$= (p_0 - c - qD)(N_0(1 + rD) + bB) - A_0 - aB. \quad (5)$$

Our goal is to maximize the profit $P$. Note that the two variables that we can control are $D$ and $B$.

Let us first consider the situation in which $B = 0$, that is, we do not consider any ad budget increase. In this case, $P$ is a function of $D$ only. We have:

$$P(D) = N_0(p_0 - c - qD)(1 + rD) - A_0. \quad (6)$$

We want to maximize $P$. Let us take the derivative. We have:

$$\frac{dP}{dD} = N_0(r(p_0 - c) - q - 2qrD). \quad (7)$$

The candidate optimal value of discount rate $D_*$ is given by:

$$\left. \frac{dP}{dD} \right|_{D = D_*} = N_0(r(p_0 - c) - q - 2qrD_*) = 0. \quad (8)$$

Thus,

$$D_* = \frac{p_0 - c}{2q} - \frac{1}{2r} = \frac{1}{4}. \quad (9)$$

It is a good idea to check the convexity of $P$ at $D_*$:

$$\left. \frac{d^2P}{dD^2} \right|_{D = D_*} = -2N_0qr < 0. \quad (10)$$

Thus, $D_*$ is at least a local maximum at $D = D_*$. In this case, $P$ is just a parabola drawn as a graph with respect to $D$, and so, $D_*$ also turns out to be the maximizer over all possible values of $D$ (global maximizer). The discount amount should be:

$$qD = 25. \quad (11)$$

So, the computer should be sold at $950 - 25 = 925$ dollars. The total profit at $D = D_*$ is given by:

$$P_* = P(D_*) = 2,481,250. \quad (12)$$

Some remarks:
• Models have limits. For example, what if I have an unlimited amount to spend for ads. Are the sales going to increase however much I increase the ad budget?

• In this problem, there was one local maximum, and this turned out to be the global maximum. Such is not always the case. In many practical applications, you can have many local maxima, and this can be a problematic.

• Optimization problems are usually solved using a numerical procedure. We shall discuss these later.

1.2 Sensitivity

Many parameters in the problem are known only to a certain degree of accuracy. We thus want to know what happens if there is some uncertainty in some of the parametric values. In the problem above $r$ may not be so reliable. Indeed, even if $r$ was the rate of increase in the test market, we are applying the same $r$ to the real market, which may be somewhat or quite different from the test market. Suppose that, instead of $r$, the rate increase in the real market is $r + \Delta r$. Then, the optimal discount rate $D_*$ would be different. Indeed, according to (9),

$$D_*(r + \Delta r) = \frac{p_0 - c}{2q} - \frac{1}{2(r + \Delta r)}.$$  \hspace{1cm} (13)

The absolute sensitivity $S_{\text{abs}}(D_*, r)$ is given by:

$$S_{\text{abs}}(D_*, r) = \frac{dD_*}{dr} = \lim_{\Delta r \to 0} \frac{D_*(r + \Delta r) - D_*(r)}{\Delta r} = \frac{1}{2r^2}. \hspace{1cm} (14)$$

The (relative) sensitivity $S(D_*, r)$ is given by:

$$S(D_*, r) = \frac{r}{D_*} \frac{dD_*}{dr} = \frac{d\log D_*}{d\log r} = \frac{1}{2rD_*}. \hspace{1cm} (15)$$

The reason one usually prefers the (relative) sensitivity to the absolute sensitivity is that the relative sensitivity does not carry any units, whereas the absolute sensitivity usually does. If we put in the numbers, we have:

$$S(D_*, r) = 4. \hspace{1cm} (16)$$
This means that, if there is a 1\% uncertainty in \( r \) (that is, \( r \) is between 0.495 and 0.505) then, there is a 4\% uncertainty in \( D_* \) (that is \( D_* \) can be between 0.24 to 0.26).

Let us now consider the sensitivity of \( P_* \) to \( r \). We must compute

\[
S(P_*, r) = \frac{r \cdot dP_*}{P_* \cdot dr}.
\]  

(17)

To find \( dP_/dr \), we must substitute the expression for \( D_* \) into \( P_* \), and take the derivative of the resulting expression in \( r \). The result is:

\[
\frac{dP_*}{dr} = N_0D_*(p_0 - c - qD_*).
\]  

(18)

We can now compute the sensitivity, and it turns out that:

\[
S(P_*, r) = 0.1134 \ldots
\]  

(19)