Asymptotics of the Airy Function

Consider the Airy function:

\[
A(x) = \int_0^\infty \cos(sx + s^3/3)ds
\]  
(1)

Here, \(-\infty < x < \infty\). We should first check that this integral is convergent. We also want to show that \(A(x)\) satisfies the differential equation:

\[
\frac{d^2A}{dx^2} -xA = 0.
\]  
(2)

We will accomplish these two tasks together.

It is difficult to deal with (1) because the integrand is oscillatory and does not decay as \(s \to \infty\). By changing the path of integration in the complex plane, we want to convert it into an integral with a rapidly decaying integrand.

Let \(\Gamma_0\) be the line \(s = \exp(\pi i/6)t, \ 0 < t < \infty\). We claim:

\[
\int_0^\infty \exp(i(sx + s^3/3))ds = \int_{\Gamma_0} \exp(i(sx + s^3/3))ds.
\]  
(3)

What we intend to show is that both of the above integrals exist, and that they are equal. Consider the right hand side of (3).

\[
\int_{\Gamma_0} \exp(i(sx + s^3/3))ds = \exp(\pi i/6) \int_0^\infty \exp(2\pi i/3)tx - t^3/3)dt.
\]  
(4)

Since the integrand above is rapidly decaying as \(t \to \infty\), it is clear that this integral exists. Thus, in order to show that the left hand side of (3) exists, we only need to show that the difference between the left and right hand side go to 0 as the end point of integration goes to infinity. Consider a point \(s = s_0 > 0 \) sufficiently large. Then,

\[
\int_{\Gamma_0, \text{Res} < s_0} \exp(i(sx + s^3/3))ds - \int_0^{s_0} \exp(i(sx + s^3/3))ds
\]  
(5)

\[
= \int_{\text{Res} = s_0, 0 < \text{arg}(s) < \pi/6} \exp(i(sx + s^3/3))ds.
\]
The last integral is along the line \( \text{Re } s = s_0, 0 < \text{Im } s < \frac{s_0}{\sqrt{3}} \).

\[
\left| \int_{\text{Re } s = s_0, 0 < \text{Im } s < \frac{s_0}{\sqrt{3}}} \exp(i(sx + s^3/3)) \, ds \right|
\]

\[
= \left| \int_0^{s_0/\sqrt{3}} \exp(i((s_0 + ir)x + (s_0 + ir)^3/3)) \, dr \right|
\]

\[
\leq \int_0^{s_0/\sqrt{3}} \exp(-rx - r(s_0^2 - r^2/3)) \, dr
\]

\[
\leq \int_0^{s_0/\sqrt{3}} \exp(-rx - r(s_0^2 - (s_0/\sqrt{3})^2/3))
\]

\[
\leq \int_0^\infty \exp(-r(x + 8s_0^2/9)) \, dr = \frac{9}{9x + 8s_0^2}
\]

In the last line, we take \( s_0 \) large enough so that \( 9x + 8s_0^2 > 0 \). As \( s_0 \to \infty \), the above integral goes to 0. Thus, we have (3). Since the left hand side of (3) exists, we have:

\[
A(x) = \text{Re } \int_0^\infty \exp(i(sx + s^3/3)) \, ds.
\]

Consider:

\[
\frac{dA}{dx} = \frac{d}{dx} \left( \exp(\pi i/6) \int_0^\infty \exp(\exp(2\pi i/3)tx - t^3/3) \, dt \right)
\]

where we used (4) to express \( A(x) \). The absolute value of derivative of the integrand in (8) is:

\[
\left| \frac{d}{dx} \exp(\exp(2\pi i/3)tx - t^3/3) \right| = \left| \exp(2\pi i/3)t \exp(\exp(2\pi i/3)tx - t^3/3) \right|
\]

\[
= t \exp(-tx/2 - t^3/3).
\]

For any \( C > 0 \) take the function:

\[
g(t) = t \exp(tC/2 - t^3/3).
\]

g(t) is clearly an integrable function over the interval \((0, \infty)\) and when \(|x| < C\),

\[
\left| \frac{d}{dx} \exp(\exp(2\pi i/3)tx - t^3/3) \right| < g(t).
\]
Thus, we may change the order of differentiation and integration in (8) as long as \(|x| < C\). But since \(C\) was arbitrary, changing the order of differentiation and integration is justified for all \(x\). Likewise, one can show that it is possible to change the order of differentiation and integration for the second derivative.

\[
\frac{d^2}{dx^2} \left( \exp(\pi i / 6) \int_0^\infty \exp(2\pi i / 3)tx - t^3 / 3)dt \right) = -i \int_0^\infty t^2 \exp(2\pi i / 3)tx - t^3 / 3)dt
\]  \hspace{1cm} (12)

Thus,

\[
\frac{d^2A}{dx^2} - xA = \text{Re} \int_0^\infty (-it^2 - \exp(\pi i / 6)x) \exp(2\pi i / 3)tx - t^3 / 3)dt
\]

\[
= \text{Im} \int_0^\infty (\exp(2\pi i / 3)x - t^2) \exp(2\pi i / 3)tx - t^3 / 3)dt
\]

\[
= \text{Im} \left( \exp(2\pi i / 3)tx - t^3 / 3) \right)_0^\infty
\]

\[
= \text{Im}(-1) = 0.
\]  \hspace{1cm} (13)

This shows (2).

We next turn to the asymptotics of the Airy function for \(|x|\) large. We look at the integral:

\[
\int_0^\infty \exp(i(sx + s^3 / 3))ds.
\]  \hspace{1cm} (14)

The saddle point of the integrand is \(s_0 = \pm \sqrt{x}i\) for \(x > 0\) and \(s_0 = \pm \sqrt{-x}\) for \(x < 0\). Since the locations of the saddle points are different for \(x > 0\) or \(x < 0\), we treat the two cases separately.

Let \(x > 0\). We rewrite (7) as:

\[
A(x) = \text{Re} \left( \frac{1}{2} \int_0^\infty \exp(i(sx + s^3 / 3))ds \right)
\]  \hspace{1cm} (15)

Let \(\Gamma_+\) be the line: \(s = \exp(-\pi i / 6)t, -\infty < t < 0\). Using the exact same calculation as (6), we see that:

\[
A(x) = \text{Re} \left( \frac{1}{2} \int_{\Gamma_+ \cup \Gamma_0} \exp(i(sx + s^3 / 3))ds \right)
\]  \hspace{1cm} (16)

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To simplify the integral, we change variables.

\[
\frac{1}{2} \int_{\Gamma_+ \cup \Gamma_0} \exp(i(sx + s^3/3))ds = \frac{\sqrt{x}}{2} \int_{\Gamma_+ \cup \Gamma_0} \exp(ix^{3/2}(t + t^3/3))dt \tag{17}
\]

The saddle point is at \( t = i \) now. The steepest descent path \( C_0 \) satisfies:

\[
\text{Im}(i(t + t^3/3)) = 0. \tag{18}
\]

This implies that \( C_0 \) is given by:

\[
b^2 - \frac{a^2}{3} = 1, \quad b > 0 \tag{19}
\]

where \( t = a + ib \). Note that the above hyperbola asymptotes to \( \Gamma_+ \) and \( \Gamma_1 \). Therefore,

\[
\frac{\sqrt{x}}{2} \int_{\Gamma_+ \cup \Gamma_0} \exp(ix^{3/2}(t + t^3/3))dt = \frac{\sqrt{x}}{2} \int_{C_0} \exp(ix^{3/2}(t + t^3/3))dt. \tag{20}
\]

We now take the following change of variables:

\[
i(t + t^3/3) = -\frac{2}{3} - (t - i)^2 + i\left(\frac{t - i}{3}\right)^3 = -\frac{2}{3} - \tau^2. \tag{21}
\]

Thus,

\[
t = i + \tau + \cdots \tag{22}
\]

We find:

\[
\frac{\sqrt{x}}{2} \int_{C_0} \exp(ix^{3/2}(t + t^3/3))dt \approx \frac{\sqrt{x}}{2} \int_{-\infty}^{\infty} \exp(-x^{3/2}(\frac{2}{3} + \tau^2))d\tau = \frac{\sqrt{\pi} \exp(-\frac{2}{3}x^{3/2})}{2x^{1/4}} \tag{23}
\]

Thus,

\[
A(x) \approx \frac{\sqrt{\pi} \exp(-\frac{2}{3}x^{3/2})}{2x^{1/4}}. \tag{24}
\]

when \( x > 0 \) and large.

Let \( x < 0 \). Consider the contour \( \Gamma_- \), \( s = it, -\infty < t < 0 \). The integral:

\[
\int_{\Gamma_-} \exp(i(sx + s^3))ds = i \int_{-\infty}^{0} \exp(-tx - t^3/3)dt \tag{25}
\]
is clearly a convergent integral, and is purely imaginary. Thus, combining this with (7), the Airy function can be written as:

\[ A(x) = \text{Re} \int_{\Gamma_- \cup \Gamma_0} \exp(i(sx + s^3/3)) ds. \]  

(26)

We will use the method of steepest descent on this integral. The integral in (26) can be written as:

\[ \int_{\Gamma_- \cup \Gamma_0} \exp(i(sx + s^3/3)) ds = \sqrt{-x} \int_{\Gamma_- \cup \Gamma_0} \exp(i(-x)^{3/2}(-s+s^3/3)) ds \]  

(27)

Let \( f(s) = i(-s + s^3/3) \). The saddle point is where \( \frac{df}{ds} = 0 \). These are points \( s_0 = \pm 1 \). The steepest descent path corresponds to:

\[ \text{Im} f(s) = \text{Im} f(s_0) \]  

(28)

At \( s_0 = 1 \), this yields:

\[ b = \pm (a - 1) \sqrt{\frac{a + 2}{3a}} \]  

(29)

where \( s = a + ib \). The curve \( C_1 : b = (a - 1) \sqrt{\frac{a + 2}{3a}} \) is the steepest descent path in this case. Note that \( C_1 \) asymptotes to \( \Gamma_- \) and \( \Gamma_0 \) (A similar calculation at \( s = -1 \) does not yield a suitable steepest descent path). Thus:

\[ \sqrt{-x} \int_{\Gamma_- \cup \Gamma_0} \exp(i(-x)^{3/2}(-s+s^3/3)) ds = \sqrt{-x} \int_{C_1} \exp(i(-x)^{3/2}(-s+s^3/3)) ds \]  

(30)

When \( |x| \) is large, the contribution to the integral is now from the point \( s_0 = 1 \). Now, set:

\[ i(-s + s^3/3) = -\frac{2}{3}i + i(s - 1)^2 + \frac{i}{3}(s - 1)^3 = -\frac{2}{3}i - \tau^2. \]  

(31)

This implies:

\[ s = 1 + \exp(\pi i/4) \tau + \cdots \]  

(32)

(one must choose \( \exp(\pi i/4) \) not \(- \exp(\pi i/4) \) because of the direction of the
(33) Thus,

\[ \int_{C_1} \exp(i(-x)^{3/2}(-s + s^3/3))ds \]
\[ \approx \sqrt{-x} \exp(\pi i/4) \int_{-\infty}^{\infty} \exp((-x)^{3/2}(-\frac{2}{3}i - \tau^2))d\tau \]
\[ = \sqrt{-x} \exp((\pi/4 - \frac{2}{3}(-x)^{3/2})i) \int_{-\infty}^{\infty} \exp((-x)^{3/2}\tau^2)d\tau \]
\[ = \sqrt{\pi} \frac{\exp((\pi/4 - \frac{2}{3}(-x)^{3/2})i)}{(-x)^{1/4}}. \]

This yields:

\[ A(x) \approx \frac{\sqrt{\pi} \cos(\frac{-\pi}{3}(-x)^{3/2} + \pi/4)}{(-x)^{1/4}} \]
\[ = \frac{\sqrt{\pi} \sin(\frac{\pi}{3}(-x)^{3/2} + \pi/4)}{(-x)^{1/4}}. \]

(34) when \( x < 0 \) and its absolute value is large.