1. Consider the following equations in $\mathbb{R}^2$:

$$\frac{\partial u}{\partial t} = \nabla \cdot (A \nabla v) = -\nabla \cdot (B \nabla w), \quad u = v - w, \quad \text{for } t > 0,$$

$$u(x, t = 0) = f(x).$$

where $u(x, t), v(x, t), w(x, t)$ are functions of $x \in \mathbb{R}^2$ and time $t$. Here, $A$ and $B$ are positive definite matrices. Solve for $u(x, t)$ using the Fourier transform in $x$. There is no need to fully invert the Fourier transforms.

2. (a) Consider the function:

$$f_\epsilon(x) = \begin{cases} \exp(-\epsilon x) & \text{if } x > 0 \\ -\exp(\epsilon x) & \text{if } x \leq 0 \end{cases}, \quad \epsilon > 0.$$  

Find the Fourier transform of the $f_\epsilon$.

(b) By taking the limit as $\epsilon \to 0$ in the above, show that the Fourier transform of the following function:

$$g(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

is given by:

$$\mathcal{F}g = \pi \delta + ih, \quad \langle h, \phi \rangle = \lim_{\epsilon \to 0} \int_{|x| > \epsilon} \frac{\phi(x)}{x} dx,$$

where $\delta$ is the Dirac delta function and $\phi$ is Schwartz function.

(c) Suppose a function $G(x)$ is a function equal to 0 when $x < 0$ (assume that the $G(x)$ is well-behaved so that the Fourier transform makes sense). Denote its Fourier transform by $\widehat{G}$. Let $\widehat{G}_r$ be the real part of $\widehat{G}$ and $\widehat{G}_i$ the imaginary part. Show that $\widehat{G}_i$ and $\widehat{G}_r$ are linked by the relations:

$$\widehat{G}_i(\xi) = (h \ast \widehat{G}_r)(\xi), \quad \widehat{G}_r(\xi) = -(h \ast \widehat{G}_i)(\xi),$$

$$\langle h \ast u \rangle(\xi) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|\xi - \eta| > \epsilon} \frac{1}{\xi - \eta} u(\eta) d\eta.$$  

Hint: Take the Fourier transform of $G = gG$. 

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MATH 8401

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3. Let \( u \) be a \( C^2 \) function in \( \mathbb{R}^2 \) satisfying:

\[
\Delta u = 0 \text{ if } r = |x| \neq 1
\]

(7)

where \( r \) is the radial coordinate. At \( r = 1 \), we assume that \( u \) is continuous and that the normal derivative satisfies the following jump condition at \( x = (\cos(\theta), \sin(\theta)) \):

\[
\left. \frac{\partial u}{\partial r} \right|_{r=1_+} - \left. \frac{\partial u}{\partial r} \right|_{r=1_-} = f(\theta)
\]

(8)

where \( f(\theta) \) is a continuous function in \( \theta \). Here, \( r = 1_\pm \) denotes the value of the derivative as one approaches \( r = 1 \) from \( r > 1 \) or \( r < 1 \).

This function \( u \) may be viewed as a distribution on \( \mathbb{R}^2 \) and satisfies:

\[
\Delta u = g
\]

(9)

where \( g \) is some distribution. Find \( g \).

4. Consider an \( n \times n \) matrix \( A \) whose column vectors are given by \( a_1, \ldots, a_n \).

(a) Suppose \( A \) is full rank and let \( A = QR \) be the QR decomposition of \( A \). Show that the length of the \( k \)-th column vector of \( R \) and the \( k \)-th column vector \( A \) are the same.

(b) Prove the inequality:

\[
|\det A| \leq \prod_{k=1}^{n} |a_k|.
\]

(10)

(this says that the volume of a parallelepiped with given side lengths is maximum when the parallelepiped is rectilinear.)

5. For a vector in \( \mathbb{C}^n \), the norm in the following is the Euclidean norm whereas for a matrix \( n \times n \) matrix \( A \), it is the norm induced by this vector norm:

\[
\| A \| = \max_{\| v \| = 1} \| Av \|.
\]

(11)

(a) Let \( \Lambda \) be a diagonal matrix with complex entries, and let \( \lambda_1, \ldots, \lambda_n \) be its eigenvalues. For \( z \in \mathbb{C} \), show that

\[
\| (zI - \Lambda)^{-1} \| = \left( \min_{k=1,...,n} |z - \lambda_k| \right)^{-1}.
\]

(12)
(b) Let $B$ be some $n \times n$ matrix. Show that $zI - (\Lambda + B)$ is invertible if
\[
\|B\| < \|(zI - \Lambda)^{-1}\|^{-1}.
\] (13)
For this, it is useful to use the following fact. For a $n \times n$ matrix $C$ satisfying $\|C\| < 1$, $(I - C)^{-1}$ exists since
\[
(I - C)^{-1} = I + C + C^2 + C^3 + \cdots,
\] (14)
and the right hand side converges thanks to $\|C\| < 1$.

(c) Show from the above that all eigenvalues of $\Lambda + B$ must lie within at least one of the $n$ circular disks in the complex plane of radius $\|B\|$ centered at $\lambda_k$.

(d) Suppose $A$ is a diagonalizable matrix such that $A = P\Lambda P^{-1}$ and $\Lambda$ is as in the above. Show that all eigenvalues of $A + B$ lie within the $n$ circular disks of the complex plane of radius $\kappa(P)\|B\|$ centered at $\lambda_k$. Here, $\kappa(P) = \|P\|\|P^{-1}\|$, the condition number of $P$.

(e) Suppose we want to compute a particular eigenvalue $\lambda \neq 0$ of a diagonalizable matrix $A$. Show that the relative condition number for this eigenvalue satisfies:
\[
\frac{|\delta\lambda|}{|\lambda|} \leq \frac{\kappa(P)\|\delta A\|}{|\lambda|},
\] (15)
where $\delta\lambda$ and $\delta A$ are the perturbations in eigenvalue $\lambda$ and the matrix $A$ respectively. In particular, for Hermetian matrices, show that $\kappa(P)$ can be taken to be 1.