Elementary Facts about Fourier Series

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1 Preliminaries: Uniform Convergence

We assume the reader is familiar with advanced calculus. We review some useful facts about uniform convergence mostly without proof. Suppose $I \subset \mathbb{R}$ be a closed interval, and consider $C(I)$, the space of continuous functions on $C(I)$. The natural norm on $C(I)$ is:

$$ f \in C(I), \|f\|_\infty = \max_{x \in I} |f(x)|. $$

(1)

Since continuous functions have a maximum on a closed interval, $\|f\|_\infty$ is finite for any function $f \in C(I)$. Suppose $f_n \in C(I), n = 1, 2, 3, \ldots$ and

$$ \forall \epsilon > 0, \exists N, \text{independent of } x \in I \text{ such that if } n \geq N, |f_n(x) - f(x)| \leq \epsilon. $$

(2)

We then say that $f_n$ converges uniformly to $f$. The above is equivalent to saying that

$$ \lim_{n \to \infty} \|f_n - f\|_\infty = 0. $$

(3)

If $f_n$ converges uniformly to $f$, then $f \in C(I)$.

Let

$$ f_n(x) = \sum_{k=1}^{n} g_k(x). $$

(4)

Suppose

$$ \lim_{n \to \infty} \sum_{k=1}^{n} |g_k(x)| < \infty. $$

(5)
Then, it is easily seen that \( f_n(x) \) will converge to some value \( f(x) \), and it is said that series converges absolutely to \( f(x) \). In this case, the order of summation does not matter. Such considerations are important when considering sums like:

\[
\sum_{k=-\infty}^{\infty} a_k.
\]

Sometimes, if we consider the two sums:

\[
\lim_{N \to \infty} \lim_{M \to -\infty} \sum_{k=M}^{N} a_k, \quad \lim_{N \to \infty} \sum_{k=-N}^{N} a_k
\]

the former may not exist but the latter may exist. Such complications will not happen if \( a_k \) is absolutely convergent. A simple test for the uniform convergence for a function series is the following (sometimes referred to as the Weierstrass \( M \)-test). If there is a sequence of real numbers \( M_k \) such that \( |g_k(x)| \leq M_k \) and

\[
\sum_{k=1}^{\infty} M_k < \infty,
\]

then the series (4) is not only absolutely convergent but also uniformly convergent (it should be noted that there are plenty of series that are uniformly but not absolutely convergent).

With regard to integration, we have the following:

**Proposition 1.** Suppose \( f_n \) converges to \( f \) uniformly. Then,

\[
\lim_{n \to \infty} \int_{I} f_n \, dx = \int_{I} f \, dx.
\]

**Proof.** Let \( |I| \) be the length of the interval.

\[
\left| \int_{I} (f_n - f) \, dx \right| \leq \int_{I} |f_n - f| \, dx \leq \|f_n - f\|_{\infty} |I| \to 0 \text{ as } n \to \infty.
\]

\[\square\]

We also take note of the following fact.

**Proposition 2.** Suppose \( f_n \in C^1(I) \), and suppose \( f_n \to f \) uniformly and \( f_n' \to g \) uniformly. Then, \( f \in C^1(I) \) and \( f' = g \).
Proof. Since $f_n \in C^1(I)$, we have, for $x_0 \in I$ and any $x \in I$,
\begin{equation}
    f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(s)ds.
\end{equation}
We may take the limit as $n \to \infty$ on both sides. By Proposition 1, we have:
\begin{equation}
    f(x) = f(x_0) + \int_{x_0}^{x} g(s)ds.
\end{equation}
Since $g \in C(I)$, $f \in C^1(I)$ and $f' = g$.

Although we have stated the above for functions of a single variable, they remain true, with suitable modifications, for functions in multiple variables.

2 Convergence of Fourier Series

Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and let $C(\mathbb{T})$ be the space of continuous functions on $\mathbb{T}$ (that is the space of continuous $2\pi$ periodic functions). We may define an inner product on this space:
\begin{equation}
    \langle f, g \rangle = \int_{0}^{2\pi} f \overline{g}dx, \ f, g \in C(\mathbb{T}),
\end{equation}
where $\overline{\cdot}$ denotes complex conjugation. It is easy to check that this satisfies the properties of an inner product. This inner product induces the following norm:
\begin{equation}
    \|f\|_2 = \sqrt{\langle f, f \rangle}.
\end{equation}
The inner product and norm above are often called the $L^2$ inner product/norm.

Now, consider the functions:
\begin{equation}
    e_k = \frac{1}{\sqrt{2\pi}} \exp(ikx), \ k \in \mathbb{Z}.
\end{equation}
These functions clearly belong to $C(\mathbb{T})$ and satisfy:
\begin{equation}
    \langle e_k, e_j \rangle = \delta_{kj} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}.
\end{equation}
This shows that the functions $e_k$ form an orthonormal set. Our experience with finite dimensional vector spaces prompts the following question. Consider the expression:

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)e_k, \quad \hat{f}(k) = \langle f, e_k \rangle.$$  

(17)

The above is called a Fourier series of $f$ and the coefficients $\langle f, e_k \rangle$ are the Fourier coefficients of $f$. The question is if and in what sense the Fourier series is equal to the original function $f$.

Before we can begin to answer this question, let us make some observations. Suppose the function $f$ belongs to $C^r(\mathbb{T})$, the space of $r$ times continuously differentiable functions that are $2\pi$ periodic. We have the following.

**Proposition 3.** Let $f \in C^r(\mathbb{T})$. Then, we have:

$$\langle f^{(r)}, e_k \rangle = (ik)^r \langle f, e_k \rangle.$$  

(18)

**Proof.** Integrate by parts. \qed

There are two important points to notice about the above result. First, it says that taking the derivative of $f$ leads to multiplication of the Fourier coefficients $\hat{f}$ with $ik$. The second observation is that, for $k \neq 0$,

$$\hat{f}(k) = (ik)^{-r} \hat{f}^{(r)}(k) = \mathcal{O}(|k|^{-r}),$$  

(19)

where $\mathcal{O}$ is the order symbol. We have used the fact that $f^r$ is a continuous function and hence that its Fourier coefficients remain bounded (this is easy to see, prove it!). It is a (very) important principle that smoothness of the function $f$ corresponds to decay properties of $\hat{f}(k)$ as $|k|$ becomes large.

Consider the following Fourier partial sum:

$$S_n f = \sum_{k=-n}^{n} \hat{f}(k)e_k.$$  

(20)

We would like to study if and in what sense $S_n f$ converges to $f$. Our first observation is the following.
Proposition 4. Let $f \in C(\mathbb{T})$. Then,
\[ \|S_n f\|_2 \leq \|f\|_2. \] (21)

Proof. First, we see that
\[ \langle S_n f, R_n f \rangle = 0, \quad R_n f = f - S_n f. \] (22)
Indeed,
\[ \langle S_n f, f \rangle = \left\langle \sum_{k=-n}^{n} \langle f, e_k \rangle e_k, f \right\rangle = \sum_{k=-n}^{n} |\langle f, e_k \rangle|^2 = \langle S_n f, S_n f \rangle. \] (23)
Thus,
\[ \langle f, f \rangle = \langle R_n f + S_n f, R_n f + S_n f \rangle = \langle R_n f, R_n f \rangle + \langle S_n f, S_n f \rangle \geq \langle S_n f, S_n f \rangle \] (24)
where we used (22) in the second equality.

Equation (22) shows that $R_n f = f - S_n f$ and $S_n f$ are orthogonal to each other. If we consider the subspace $S_n$ of $C(\mathbb{T})$ spanned by the functions $e_k, -n \leq k \leq n$, then $S_n f$ is the orthogonal projection of $f$ onto $S_n$.

Proposition 5 (Bessel’s Inequality). Suppose $f \in C(\mathbb{T})$. Then we have:
\[ \sum_{k=-\infty}^{\infty} |\langle f, e_k \rangle|^2 \leq \|f\|_2^2. \] (25)

Proof. By Proposition 4, we have:
\[ \sum_{k=-n}^{n} |\langle f, e_k \rangle|^2 = \|S_n f\|_2^2 \leq \|f\|_2^2. \] (26)
Take the limit as $n \to \infty$.

As a corollary, we have:

Corollary 1. Suppose $f \in C(\mathbb{T})$. We have
\[ \lim_{k \to \pm \infty} \hat{f}(k) = 0. \] (27)
Proof. This must be so since the sum of $\left|\tilde{f}(k)\right|^2$ is finite according to the Bessel’s inequality.

Note that the conclusions of Proposition 4, 5 and Corollary (1) are also true for any function whose $L^2$ norm is finite. For instance, these results hold for any piecewise continuous function.

We are now ready to prove our first result on convergence of Fourier Series. For $f \in C(T)$, let

$$\|f\|_\infty = \max_{x \in T} |f(x)|.$$  \hfill (28)

**Theorem 1.** Suppose $f \in C^1(T)$. Then, $S_n f$ converges uniformly to $f$. Equivalently,

$$\lim_{n \to \infty} \|S_n f - f\|_\infty = 0.$$  \hfill (29)

**Proof.** We first write $S_n f$ in the following way:

$$(S_n f)(x) = \sum_{k=-n}^{n} \langle f, e_k \rangle e_k$$

$$= \sum_{k=-n}^{n} \frac{1}{2\pi} \left( \int_{0}^{2\pi} f(y) \exp(-iky)dy \right) \exp(ikx)$$

$$= \int_{0}^{2\pi} f(y) D_n(x-y)dy = \int_{0}^{2\pi} f(x-y)D_n(y)dy$$  \hfill (30)

where

$$D_n(y) = \frac{1}{2\pi} \sum_{k=-n}^{n} \exp(iky) = \frac{\sin((n + 1/2)y)}{2\pi \sin(y/2)}.$$  \hfill (31)

The function $D_n(y)$ is known as the *Dirichlet kernel*. Now, note that

$$\int_{0}^{2\pi} D_n(y)dy = \int_{0}^{2\pi} \left( \frac{1}{2\pi} \sum_{k=-n}^{n} \exp(iky) \right) dy = 1.$$  \hfill (32)

Thus,

$$f(x) - (S_n f)(x) = \int_{0}^{2\pi} (f(x) - f(x-y))D_n(y)dy$$

$$= \int_{0}^{2\pi} g_x(y) \sin((n + 1/2)y)dy$$  \hfill (33)
where
\[ g_x(y) = \frac{f(x) - f(x - y)}{2\pi \sin(y/2)} \] (34)

The function \( g_x(y) \) is a continuous function for \( y \neq 0 \). It is also continuous at \( y = 0 \) since:
\[ \int_0^{2\pi} g_x(y) \sin((n + 1/2)y) dy = \lim_{y \to 0} g_x(y) = \frac{1}{\pi} f'(x). \] (35)

Here we used our assumption that \( f \) is a \( C^1 \) function. The last integral in (33) can thus be written as:
\[ \int_0^{2\pi} g_x(y) \sin((n + 1/2)y) dy = \frac{1}{2i} \int_0^{2\pi} (g_x(y) \exp((n + 1/2)iy) - g_x(y) \exp(-i(n + 1/2)y)) dy \] (36)

By Corollary 1, the last line above tends to 0 as \( n \to \infty \) (note that the functions \( g_x(y) \exp(\pm iy/2) \) are only piecewise continuous in \( y \), but Corollary 1 is true for such functions as well, as discussed after its proof). Equation (33) thus ends to 0 and \( (S_n f)(x) \) converges to \( f(x) \) for every \( x \).

We must now prove that the convergence is uniform. To do so, we estimate the difference \( S_n f - S_m f \) for \( m > n \).

\[ |(S_n f)(x) - (S_m f)(x)| = \left| \sum_{n < |k| \leq m} \hat{f}(k) \frac{\exp(ikx)}{\sqrt{2\pi}} \right| \leq \frac{1}{\sqrt{2\pi}} \sum_{n < |k| \leq m} \left| \hat{f}(k) \right| \] (37)

Now, since \( f \in C^1(\mathbb{T}) \), by Proposition 3 we have:
\[ \langle f, e_k \rangle = \frac{1}{ik} \langle f', e_k \rangle. \] (38)

We have, therefore,
\[ \frac{1}{\sqrt{2\pi}} \sum_{n < |k| \leq m} |\hat{f}(k)| \leq \frac{1}{\sqrt{2\pi}} \left( \sum_{n < |k| \leq m} \frac{1}{k^2} \right)^{1/2} \left( \sum_{n < |k| \leq m} |\hat{f}'(k)|^2 \right)^{1/2} \] (39)

\[ \leq \frac{2}{\sqrt{2\pi n}} \left( \sum_{|k| > n} |\hat{f}'(k)|^2 \right)^{1/2} \leq \frac{2}{\sqrt{2\pi n}} \|f'\|_2 \]
where we used the Cauchy-Schwarz inequality in the first inequality and Bessel’s inequality in the last inequality. We thus have:

$$|(S_n f)(x) - (S_m f)(x)| \leq \frac{2}{\sqrt{2\pi n}} \|f''\|_2. \quad (40)$$

Taking the limit $m \to \infty$ and taking the maximum in $x$, we have:

$$\|S_n f - f\|_{\infty} \leq \frac{2}{\sqrt{2\pi n}} \|f''\|_2. \quad (41)$$

Uniform convergence is now immediate.

We make the following observation, which is of some interest.

**Proposition 6.** Suppose $f \in L^2(\mathbb{T})$ is continuous at $x \in \mathbb{T}$ and

$$g_x(y) = \frac{f(x) - f(x - y)}{2\pi \sin(y/2)} \in L^2(\mathbb{T}). \quad (42)$$

Then, $(S_n f)(x) \to f(x)$ as $n \to \infty$.

The proof is identical to the proof in Theorem 1.

A slight modification of the uniform convergence argument in the proof of Theorem 1 gives us the following.

**Proposition 7.** Let $f \in C^r(\mathbb{T})$, $r \geq 1$. Then,

$$\lim_{n \to \infty} n^{r-1/2} \|S_n f - f\|_{\infty} = 0. \quad (43)$$

**Proof.** Assuming $f \in C^r(\mathbb{T})$, a calculation parallel to the one given in (37)-(39) gives

$$|(S_n f)(x) - (S_m f)(x)| \leq \frac{2}{\sqrt{2\pi (2r - 1)n^{r-1/2}}} \left( \sum_{|k| > n} \left| \hat{f}^{(r)}(k) \right|^2 \right)^{1/2}. \quad (44)$$

Taking the limit as $m \to \infty$ and the maximum in $x$, we find that

$$n^{r-1/2} \|S_n f - f\|_{\infty} \leq \frac{2}{\sqrt{2\pi (2r - 1)}} \left( \sum_{|k| > n} \left| \hat{f}^{(r)}(k) \right|^2 \right)^{1/2}. \quad (45)$$

The right hand side of the above inequality tends to 0 as $n \to \infty$ thanks to Bessel’s inequality.
The above proposition shows that smoothness of $f$ translates to faster convergence of the Fourier series. This is consequence of the fact that the Fourier coefficients decay rapidly when the function is smooth. It is a general phenomenon in functional approximation (or numerical methods) that approximations to smooth functions converge more rapidly.

We now know that, if $f$ continuously differentiable, the Fourier series $S_n f$ converges to $f$. What if $f$ is merely continuous?

**Theorem 2.** Suppose $f \in C(\mathbb{T})$. Then, $S_n f$ converges to $f$ in the $L^2$ norm:

$$\lim_{n \to \infty} \|S_n f - f\|_2 = 0. \quad (46)$$

We first note that convergence in the max norm implies convergence in the $L^2$ norm since, for $f \in C(\mathbb{T})$,

$$\|f\|_2^2 = \int_0^{2\pi} |f(x)|^2 \, dx \leq \int_0^{2\pi} \|f\|_\infty^2 \, dx = 2\pi \|f\|_\infty^2. \quad (47)$$

So the above theorem is certainly true for $f \in C^1(\mathbb{T})$. To show that the above is valid also for $f \in C(\mathbb{T})$, we first show that any function in $C(\mathbb{T})$ can be approximated arbitrarily closely by a $C^1(\mathbb{T})$ function in the $L^2$ norm. In fact, what we will show is considerably stronger:

**Proposition 8.** Let $f \in C(\mathbb{T})$. For any $\epsilon > 0$, there is a function $g \in C^\infty(\mathbb{T})$ such that

$$\|f - g\|_\infty \leq \epsilon. \quad (48)$$

**Proof.** Let $\phi$ be any function in $C^\infty(\mathbb{T})$ and $f \in C(\mathbb{T})$. We first show that the function

$$g(x) = \int_0^{2\pi} f(y) \phi(x - y) \, dy \quad (49)$$

is in $C^1(\mathbb{T})$. Consider the following difference:

$$g(x + h) - g(x) = \int_0^{2\pi} f(y) (\phi(x + h - y) - \phi(x - y)) \, dy$$

$$= \int_0^{2\pi} f(y) \left( \int_x^{x+h} \phi'(z - y) \, dz \right) \, dy$$

$$= h \int_0^{2\pi} f(y) \left( \int_0^1 \phi'(x + sh - y) \, ds \right) \, dy. \quad (50)$$
For any $\epsilon > 0$, there is a $\delta > 0$ (independent of $x$ or $y$) so that for $|h| \leq \delta$, so that
\[
\left| \int_0^1 \phi'(x + sh - y)ds - \phi'(x - y) \right| \leq \epsilon 
\] (51)
This comes from the uniform continuity of $\phi'$. Let
\[
q(x) = \int_0^{2\pi} f(y)\phi'(x - y)dy.
\] (52)
Combining (50)-(52), we have:
\[
|h^{-1}(g(x + h) - g(x)) - q(x)| \leq \int_0^{2\pi} |f(y)| \epsilon dy \leq 2\pi \|f\|_{\infty} \epsilon
\] (53)
for $|h| \leq \delta$. This shows that
\[
\lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = q(x)
\] (54)
and since $q(x)$ is a continuous function (by uniform continuity of $\phi'$), $g(x) \in C^1(T)$. By applying the same argument to $q(x)$, we see that $g(x)$ is in $C^1$ and hence that $g(x)$ is in $C^2$. By repeating this argument, we conclude that $g$ is $C^\infty$.

Consider a non-negative $C^\infty$ function $\phi$ that is 0 for $|x| \geq 1$ and satisfies:
\[
\int_{-1}^1 \phi(x)dx = 1.
\] (55)
Now, let
\[
\phi_h(x) = \frac{1}{h} \phi \left( \frac{x}{h} \right).
\] (56)
Define
\[
f_\delta(x) = \int_0^{2\pi} f(y)\phi_\delta(x - y)dy.
\] (57)
We know that $f_\delta$ is a $C^\infty$ function. Let us estimate the difference between $f_\delta$ and $f$.
\[
|f_\delta(x) - f(x)| \leq \left| \int_0^{2\pi} f(y)\phi_\delta(x - y)dy - f(x) \right|
\leq \int_0^{2\pi} |f(x - y) - f(x)| \phi_\delta(y)dy
\leq \int_{|y| \leq \delta} |f(x - y) - f(x)| \phi_\delta(y)dy
\] (58)
where we used (55) and changed variables in the second inequality. By uniform continuity of $f$, we may take $\delta$ small enough so that the difference $|f(x - y) - f(x)| \leq \epsilon$ for any choice of $x$ so long as $|y| \leq \delta$. Thus,

$$|f_\delta(x) - f(x)| \leq \int_{|y| \leq \delta} \epsilon \phi_\delta(y) dy = \epsilon$$

(59)

where we used (55) in the last equality. Taking the maximum in $x$, we have

$$\|f_\delta - f\|_\infty \leq \epsilon.$$  

(60)

Corollary 2. Let $f \in C(\mathbb{T})$. For any $\epsilon > 0$, there is a function $g \in C^\infty(\mathbb{T})$ such that

$$\|f - g\|_2 \leq \epsilon.$$  

(61)

Proof. Take any $\epsilon > 0$ From Proposition 8, there is a function $g \in C^\infty(\mathbb{T})$ such that

$$\|f - g\|_\infty \leq \frac{\epsilon}{\sqrt{2\pi}}.$$  

(62)

From (47), we have

$$\|f - g\|_2 \leq \sqrt{2\pi} \|f - g\|_\infty \leq \epsilon.$$  

(63)

In fact, that above is true for any function in $L^2(\mathbb{T})$. We shall not need this fact.

We are now ready to prove Theorem 2.

Proof of Theorem 2. Let $f \in C(\mathbb{T})$ and a $\epsilon > 0$. According to Corollary 2, there is a $g \in C^1(\mathbb{T})$ be such that

$$\|f - g\|_2 \leq \epsilon.$$  

(64)

Let us estimate the difference between $S_n f$ and $f$:

$$\|S_n f - f\|_2 \leq \|S_n (f - g)\|_2 + \|S_n g - g\|_2 + \|f - g\|_2 \leq 2 \|f - g\|_2 + \|S_n g - g\|_2 \leq 2\epsilon + \|S_n g - g\|_2.$$  

(65)
where we used Proposition 4 in the second inequality and (64) in the last inequality. Now, $S_n g$ converges to $g$ in the max norm and hence in the $L^2$ norm (by the remark right after the statement of Theorem 2). Thus, for sufficiently large $n$, we have:

$$\|S_n f - f\|_2 \leq 2\varepsilon + \|S_n g - g\|_2 \leq 3\varepsilon.$$  \hspace{1cm} (66)

This was what was to be proved.

One may naturally wonder whether $S_n f$ converges to $f$ uniformly (in the max norm) for any $f \in C(T)$. The answer is no. The difficulty can be seen as follows. Suppose we try to prove that $S_n f$ converges to $f$ using the same strategy as in the proof above of Theorem 2. The proof of Proposition 8 gives a $g \in C^1(T)$ such that

$$\|f - g\|_\infty \leq \varepsilon.$$  \hspace{1cm} (67)

Therefore,

$$\|S_n f - f\|_\infty \leq \|S_n (f - g)\|_\infty + \|S_n g - g\|_\infty + \|f - g\|_\infty.$$  \hspace{1cm} (68)

The second and last terms can be controlled. The problem is with the first term. Recall that, in the case of the $L^2$ norm, the crucial estimate was supplied by Proposition 4 which states that $\|S_n f\|$ is always smaller than $\|f\|$. The question then is the following. Is there a $K > 0$ such that

$$\|S_n f\|_\infty \leq K \|f\|_\infty.$$  \hspace{1cm} (69)

where $K$ is independent of $n$? If such a $K$ exists, the we can prove that $S_n f$ converges to $f$ in the max norm. The answer, unfortunately, is no. Define the operator norm of $S_n$ in the max norm as:

$$\|S_n\|_\infty = \sup_{\|f\|_\infty = 1} \|S_n f\|_\infty.$$  \hspace{1cm} (70)

It is not difficult to see that

$$\|S_n\|_\infty = \int_0^{2\pi} |D_n(y)| dy, \quad D_n(y) = \frac{\sin((n + 1/2)y)}{2\pi \sin(y/2)},$$  \hspace{1cm} (71)

where $D_n$ is the Dirichlet kernel (31) introduced in the proof of Theorem 1. This is an oscillatory function, and it can be easily checked that $\|S_n\|_\infty$
grows like \( \log n \) as \( n \) becomes large. This means that there is no \( K \) satisfying (69). That is to say:

\[
\sup_{n \in \mathbb{N}} \| S_n \|_\infty = \infty. \tag{72}
\]

Mimicking the proof of Theorem 2 for the max norm will therefore not work.

In fact, combined with an abstract result from functional analysis (the uniform boundedness principle) (72) is enough to conclude that there are continuous functions for which \( S_n f \) does not converge to \( f \) uniformly.

There are ways to sum the Fourier series to overcome this difficulty. One such way is to use Féjer summation:

\[
(T_n f)(x) = \sum_{k=-n}^{n} \left( 1 - \frac{|k|}{n} \right) \hat{f}(k)e_k. \tag{73}
\]

Here, we are weighting the Fourier coefficients so that it does not get abruptly cut off at \( |k| = n \). It can be shown that the Féjer sum converges uniformly to the function \( f \).

As a final topic in relation to the convergence of Fourier series, we consider the Gibbs phenomenon. Let us consider the function:

\[
\phi(x) = \begin{cases} 
\pi - x & \text{if } 0 < x \leq \pi \\
0 & \text{if } x = 0 \\
-\pi - x & \text{if } -\pi \leq x < 0 
\end{cases} \tag{74}
\]

We would like to know if \( S_n \phi \) converges to \( \phi \). This function is not in \( C(\mathbb{T}) \) so the foregoing theory does not tell us whether we will get convergence (in fact, it is not difficult to see that \( \phi \) can be approximated arbitrarily closely by a \( C^1(\mathbb{T}) \) function in the \( L^2 \) norm, so convergence in the \( L^2 \) norm is easily seen). To do so, we first compute the Fourier coefficients explicitly. We find that the Fourier partial sum is given by:

\[
(S_n \phi)(x) = \sum_{|k| \leq n, k \neq 0} \frac{1}{ik} \exp(ikx). \tag{75}
\]

Comparing this expression to (31), we have:

\[
(S_n \phi)(x) = \int_0^x 2\pi D_n(y) dy - x = \int_0^x \sin((n + 1/2)y) \sin(y/2) dy - x. \tag{76}
\]
Proposition 6 shows that

\[
\lim_{n \to \infty} S_n \phi(x) = \begin{cases} 
\phi(x) & \text{if } x \neq 0, \\
0 & \text{if } x = 0.
\end{cases}
\]

(77)

The convergence, however, cannot be uniform. Indeed, if the convergence were uniform, \( \phi \) would be continuous (since \( S_n \phi \) is continuous). To understand how uniform convergence fails, we consider the difference:

\[
S_n \phi(x) - \phi(x) = \int_0^x \frac{\sin((n + 1/2)y)}{\sin(y/2)} dy - \pi, \text{ where } x > 0.
\]

(78)

The maximum of this quantity is clearly attained at \( x = \pi/(n + 1/2) \). Let us evaluate this value as \( n \to \infty \):

\[
\lim_{n \to \infty} (S_n \phi(\pi/(n + 1/2)) - \phi(\pi/(n + 1/2)))
= \lim_{n \to \infty} \int_0^{\pi/(n+1/2)} \frac{\sin((n + 1/2)y)}{\sin(y/2)} dy - \pi
= \lim_{n \to \infty} \int_0^{\pi} \frac{\sin w}{(n + 1/2) \sin(w/(2n + 1))} dw - \pi
= 2 \int_0^{\pi} \frac{\sin w}{w} dw - \pi.
\]

(79)

The last equality requires justification. We must make sure that we can interchange the integral and the limit, but we omit this argument (it is not difficult; you are invited to justify this). It turns out that:

\[
\frac{2}{\pi} \int_0^{\pi} \frac{\sin w}{w} dw - 1 = 0.178979744 \cdots
\]

(80)

Thus, there is a “jump” of about 18% with respect to half the magnitude of the discontinuity at 0. This is known as the Gibbs phenomenon.

This is in fact a general phenomenon for piecewise \( C^1 \) functions, and not just for the function \( \phi \). Suppose we have a function \( f \) defined on \( \mathbb{T} \) such that it has jump discontinuities at \( x = x_1, \cdots, x_n \), and at each discontinuity, we have:

\[
f(x_k+) - f(x_k-) = a_k, \quad k = 1, \cdots, n
\]

(81)

where \( x_k\pm \) denote the limiting values of \( f \) as \( x \) approaches \( x_k \) from above or below. In between these discontinuities, \( f(x) \) is a \( C^1 \) function (up to the discontinuities). Now, consider the function:

\[
g(x) = f(x) - \sum_{k=1}^n \frac{a_k}{2\pi} \phi(x - x_k).
\]

(82)
Then, \( g(x) \) is a continuous and piecewise \( C^1 \) function. Using the same argument as in the proof of Theorem 1, it is easily seen that \( S_n g \) converges uniformly to \( g \). Thus, the Gibbs phenomenon with the same 18% jump is seen at every discontinuity.

# 3 Some Applications

## 3.1 Numerical Integration

Consider a \( f \in C^r(\mathbb{T}) \). We would like to evaluate the integral:

\[
\int_0^{2\pi} f dx.
\]

(83)

A simple approximation to this integral would be the following:

\[
I_N(f) \equiv \sum_{l=0}^{N-1} f(x_l) \Delta x, \; x_l = l \Delta x, \; \Delta x = \frac{2\pi}{N}.
\]

(84)

For a function that is not periodic defined on an interval, this approximation of the integral is only first order accurate in the sense that, if \( f \) is a \( C^r \) \((r \geq 1)\) function,

\[
\left| \int_0^{2\pi} f dx - I_N(f) \right| \leq C \Delta x = \frac{2\pi C}{N}
\]

(85)

where \( C \) is a constant that only depends on \( f \). However, for \( 2\pi \) periodic functions, the following estimate holds.

**Proposition 9.** Let \( f \in C^r(\mathbb{T}), r \geq 1 \). Then,

\[
\left| \int_0^{2\pi} f dx - I_N(f) \right| \leq C_r \| f^{(r)} \|_2 N^{-r}
\]

(86)

where the constant \( C_r \) depends only on \( r \).

**Proof.** First, note that,

\[
\int_0^{2\pi} f dx = \sqrt{2\pi} \int_0^{2\pi} f \cdot \frac{1}{\sqrt{2\pi}} dx = \sqrt{2\pi} \langle f, e_0 \rangle.
\]

(87)
We now evaluate $I_N(f)$, expanding $f$ in terms of Fourier series. Consider:

$$S_n f = \sum_{k=-n}^{n} \langle f, e_k \rangle e_k. \tag{88}$$

We have:

$$I_N(S_n f) = \sum_{k=-n}^{n} \langle f, e_k \rangle I_N(e_k). \tag{89}$$

Let us evaluate $I_N(e_k)$.

$$I_N(e_k) = \frac{1}{\sqrt{2\pi}} \sum_{l=0}^{N-1} \left( \exp \left( \frac{2\pi ik}{N} \right) \right) \frac{2\pi}{N} = \begin{cases} \sqrt{2\pi} & \text{if } k \in N\mathbb{Z}, \\ 0 & \text{if } k \notin N\mathbb{Z}. \end{cases} \tag{90}$$

Thus,

$$I_N(S_n f) = \sqrt{2\pi} \sum_{|j| \leq n} \langle f, e_j N \rangle \tag{91}$$

Since $f \in C^1(\mathbb{T})$, $S_n f$ converges uniformly to $f$ (see Theorem 1). Thus,

$$I_N(f) = \lim_{n \to \infty} I_N(S_n f) = \lim_{n \to \infty} \sqrt{2\pi} \sum_{|j| \leq n} \langle f, e_j N \rangle \tag{92}$$

Using (87), we have:

$$\left| \int_0^{2\pi} f dx - I_N(f) \right| \leq \lim_{n \to \infty} \sqrt{2\pi} \sum_{1 \leq |j| \leq n} |\langle f, e_j N \rangle|$$

$$= \lim_{n \to \infty} \sqrt{2\pi} \sum_{1 \leq |j| \leq n} \left| \left\langle f^{(r)}, e_j N \right\rangle \right| |jN|^{-r}$$

$$\leq \sqrt{2\pi} \left( \sum_{j \in \mathbb{Z}} \left| \left\langle f^{(r)}, e_j N \right\rangle \right|^2 \right)^{1/2} \left( 2 \sum_{j \in \mathbb{N}} |j|^{-2r} \right)^{1/2} N^{-r} \tag{93}$$

$$\leq C_r \| f^{(r)} \|_2 N^{-r},$$

where we used Proposition 3 in the equality above, and the Cauchy Schwartz inequality in the second inequality.

The above result says that, if $f \in C^\infty(\mathbb{T})$, then $I_N(f)$ converges to the integral at a rate faster than any power of $1/N$. This property is called **spectral accuracy**.

---

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We now turn to a completely different way of computing the integral. Let 
\( f \in C(\mathbb{T}) \). Our goal is still to compute the integral (83). We consider the 
following scheme. Let:
\[
x_l = x_0 + 2\pi \alpha l, \quad \alpha : \text{irrational}.
\]
(94)

We consider:
\[
J_N(f) = \frac{1}{N} \sum_{l=0}^{N-1} f(x_l).
\]
(95)

What we are doing here is that we are sampling the value of \( f \) at different 
points and taking the average. It is important here that \( \alpha \) is irrational. 
Otherwise, \( x_l = x_0 \) at some \( l > 0 \), and we will only be able to sample \( f \) at 
a finite number of points.

**Proposition 10.** Suppose \( f \in C(\mathbb{T}) \). Then,
\[
\lim_{N \to \infty} J_N(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx.
\]
(96)

**Proof.** Let us first prove the assertion for a finite sum of complex exponen-
tials:
\[
p(x) = \sum_{|k| \leq n} c_k e_k(x)
\]
(97)

where \( c_k \) are constants. Note first that
\[
\frac{1}{2\pi} \int_0^{2\pi} p(x) \, dx = \frac{1}{\sqrt{2\pi}} c_0.
\]
(98)

Let us evaluate \( J_N(p) \).
\[
J_N(p) = \sum_{|k| \leq n} c_k J_N(e_k).
\]
(99)

It is easily seen that
\[
J_N(e_0) = \frac{1}{\sqrt{2\pi}}.
\]
(100)

For \( k \neq 0 \), we have:
\[
J_N(e_k) = \frac{1}{\sqrt{2\pi N}} \sum_{l=0}^{N} \exp(ikx_0)(\exp(2\pi i\alpha))^l
\]
\[
= \frac{1}{\sqrt{2\pi N}} \exp(ikx_0) \frac{1 - \exp(2\pi i\alpha)}{1 - \exp(2\pi i\alpha)}.
\]
(101)
It is in the second equality above that we use the irrationality of $\alpha$. Since $\alpha$ is irrational, $k\alpha$ is not an integer for any integer $k$, and thus $\exp(2\pi ik\alpha) \neq 1$. Thus,

$$\left| \frac{1}{2\pi} \int_0^{2\pi} p\,dx - J_N(p) \right| \leq \sum_{1 \leq |k| \leq n} |c_k| |J_N(e_k)|$$

$$\leq \frac{1}{\sqrt{2\pi N}} \sum_{1 \leq |k| \leq n} \frac{2|c_k|}{|1 - \exp(2\pi ik\alpha)|} \to 0 \text{ as } N \to \infty. \quad (102)$$

We will now prove our desired result for $f \in C(\mathbb{T})$. Recall from Proposition 8 that there is a $g \in C^1(\mathbb{T})$ such that

$$\|f - g\|_\infty \leq \epsilon. \quad (103)$$

By Theorem 1, by taking $n$ large enough, $p = S_n g$ satisfies:

$$\|g - p\|_\infty \leq \epsilon. \quad (104)$$

Combining the above, we have:

$$\|f - p\|_\infty \leq \|f - g\|_\infty + \|g - p\|_\infty \leq 2\epsilon. \quad (105)$$

Note that $p$ is a finite sum of complex exponentials. Thus,

$$\left| \frac{1}{2\pi} \int_0^{2\pi} f\,dx - J_N(f) \right|$$

$$\leq \left| \frac{1}{2\pi} \int_0^{2\pi} (f - p)\,dx \right| + \left| \frac{1}{2\pi} \int_0^{2\pi} p\,dx - J_N(p) \right| + |J_N(f - p)|$$

$$\leq 2\|f - p\|_\infty + \frac{1}{2\pi} \int_0^{2\pi} p\,dx - J_N(p) \leq 2\epsilon + \frac{1}{2\pi} \int_0^{2\pi} p\,dx - J_N(p). \quad (106)$$

By (102), we have the desired result. \hfill \Box

What we have above can be thought of as a kind of ergodic theorem. If we think of $f(x_l)$ as measurements of a function at time $l$, the above statement is saying that the “time average” is equal to the “phase average”.
3.2 Heat Equation

Solving the heat equation was Fourier’s original motivation for introducing Fourier series. Consider the equation:

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{for } t > 0, \ x \in \mathbb{T} = \mathbb{R}/(2\pi \mathbb{Z}),
\]

\[
u(x,0) = f(x), \ f \in C(\mathbb{T})
\] (107)

We proceed as follows. Assume that \(u(x,t)\) have the following expansion:

\[
u(x,t) = \sum_{k=-\infty}^{\infty} u_k(t)e_k(x)
\] (108)

where \(e_k\) is as in (15). Not worrying about convergence issues for the moment, take the derivative of the above expression with respect to \(t\). We have:

\[
\frac{\partial u}{\partial t} = \sum_{k=-\infty}^{\infty} \frac{du_k}{dt} e_k.
\] (109)

Likewise, we have:

\[
\frac{\partial^2 u}{\partial x^2} = -\sum_{k=-\infty}^{\infty} k^2 u_k e_k.
\] (110)

Since \(u\) satisfies the heat equation, we must have:

\[
\sum_{k=-\infty}^{\infty} \frac{du_k}{dt} e_k = -\sum_{k=-\infty}^{\infty} k^2 u_k e_k.
\] (111)

Since all Fourier coefficients must be the same (or equivalently, take the inner product with respect to \(e_l\) on both sides):

\[
\frac{du_k}{dt} = -k^2 u_k.
\] (112)

Thus,

\[
u(x,t) = \sum_{k=-\infty}^{\infty} c_k \exp(-k^2 t)e_k(x).
\] (113)

Now, if we let \(t = 0\) in the above,

\[
f(x) = u(x,0) = \sum_{k=-\infty}^{\infty} c_k e_k(x).
\] (114)
Therefore, $c_k$ are the Fourier coefficients of $f(x)$ and

$$u(x, t) = (H_t f)(x) \equiv \sum_{k=-\infty}^{\infty} \langle f, e_k \rangle \exp(-k^2 t) e_k(x). \quad (115)$$

We now want to justify this claim. First, note that expression (115) makes sense for $t > 0$. By this, we mean that the sum (115) is absolutely convergent. To see this, note that:

$$|u(x, t)| \leq \sum_{k=-\infty}^{\infty} C \exp(-k^2 t) \frac{1}{\sqrt{2\pi}} < \infty, \quad |\langle f, e_k \rangle| \leq C \quad (116)$$

where we used that fact that $\langle f, e_k \rangle$ is bounded by a constant $C$ independent of $k$ (since $f$ is in $C(T)$). In fact, by the Weierstrass $M$-test, it is clear that the series converges uniformly for $(x, t) \in T \times [\epsilon, \infty)$ for any $\epsilon > 0$. Thus, the expression (115) is continuous in time and space for $t > 0$. In fact, we have the following:

**Proposition 11.** For $t > 0$, the expression $u(x, t)$ of (115) is $C^\infty$ in $(x, t)$ (can be differentiated in $x$ and $t$ arbitrarily many times) and the derivative is given by:

$$\frac{\partial^m}{\partial t^m} \frac{\partial^l}{\partial x^l} u(x, t) = \sum_{k=-\infty}^{\infty} (-k^2)^m (ik)^l \langle f, e_k \rangle \exp(-k^2 t) e_k(x). \quad (117)$$

**Proof.** We will prove the case just for $l = 1$ and $m = 0$.

$$u_n = \sum_{k=-n}^{n} \langle f, e_k \rangle \exp(-k^2 t) e_k(x), \quad \frac{\partial u_n}{\partial x} = v_n = \sum_{k=-n}^{n} ik \langle f, e_k \rangle \exp(-k^2 t) e_k(x). \quad (118)$$

For $(x, t) \in T \times [\epsilon, \infty)$ where $\epsilon > 0$, both $u_n$ and $v_n$ are uniformly and absolutely convergent by the Weierstrass $M$-test, and $u_n$ converges to $u$ and $v_n$ to:

$$\lim_{n \to \infty} v_n = v = \sum_{k=-\infty}^{\infty} ik \langle f, e_k \rangle \exp(-k^2 t) e_k(x). \quad (119)$$

By Proposition 2 (strictly speaking, this Proposition must be adapted so that it applies to multivariable functions; this is very easy), we see that
\[ \frac{\partial u}{\partial x} = v. \] The general case for larger values of \( l \) and \( m \) can be obtained by a repeated application of this argument.

As a direct consequence of this, we have:

**Corollary 3.** For \( t > 0 \), \( u \) given in (115) satisfies:

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \] (120)

The crucial thing that made the above assertions possible was the factor \( \exp(-k^2t) \) multiplying the Fourier coefficients. This made summation possible even when the derivative is taken so that arbitrary powers of \( k \) get multiplied to the coefficients.

Although we now know that \( u \) of (115) indeed satisfies the heat equation for \( t > 0 \), we do not know what happens as \( t \to 0 \). Our expectation is that

\[ \lim_{t \to 0} u(x,t) = f(x). \] (121)

In fact, this convergence is uniform if \( f \in C(\mathbb{T}) \).

**Theorem 3.** Suppose \( f \in C(\mathbb{T}) \). Then,

\[ \lim_{t \to 0} \| H_t f - f \|_{\infty} = 0. \] (122)

The proof of this statement will come later. As in the convergence proof for Fourier series, we first prove this assertion when \( f \) is sufficiently smooth.

**Proposition 12.** The statement (122) is true for \( f \in C^2(\mathbb{T}) \).

**Proof.** Since \( f \in C^2(\mathbb{T}) \),

\[ |\langle f, e_k \rangle| = \left| \frac{\langle f^{(2)}, e_k \rangle}{k^2} \right| \leq \frac{C}{k^2}, k \neq 0 \] (123)

by Proposition 3. Thus, the Fourier series:

\[ f = \sum_{k=-\infty}^{\infty} \langle f, e_k \rangle e_k \] (124)
is absolutely and uniformly convergent. Now,

\[ |(H_t f)(x) - f(x)| = \left| \sum_{k=-\infty}^{\infty} \langle f, e_k \rangle (1 - \exp(-tk^2))e_k \right| \]

\[ \leq \sum_{|k| \geq 1} \frac{C}{\sqrt{2\pi k^2}} (1 - \exp(-tk^2)), \]

where we used (123) in the above inequality. If suffices to show that the last sum tends to 0 as \( t \to 0 \). Take an arbitrary \( \epsilon > 0 \) and, take \( N \) so large that

\[ \frac{C}{\sqrt{2\pi}} \sum_{|k| > N} \frac{1}{k^2} \leq \frac{\epsilon}{2}. \]

Then,

\[ \sum_{|k| \geq 1} \frac{C}{\sqrt{2\pi k^2}} (1 - \exp(-tk^2)) \leq \epsilon + \sum_{1 \leq |k| \leq N} \frac{C}{\sqrt{2\pi k^2}} (1 - \exp(-tk^2)). \]

The sum on the right hand side is a finite sum, and clearly tends to 0 as \( t \to 0 \). \( \square \)

To prove Theorem 3, we first make an excursion to rewrite \( u(t, x) = (H_t f)(x) \) in a different way.

\[ (H_t f)(x) = \sum_{k=-\infty}^{\infty} \langle f, e_k \rangle \exp(-tk^2)e_k(x) \]

\[ = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{0}^{2\pi} f(y) \exp(-iky)dy \exp(ikx - tk^2) \]

\[ = \frac{1}{2\pi} \int_{0}^{2\pi} \sum_{k=-\infty}^{\infty} \exp(-tk^2 + ik(x - y))f(y)dy \]

\[ = \int_{0}^{2\pi} G_t(x - y)f(y)dy, \]

where

\[ G_t(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \exp(-tk^2 + ikx) \]

In equation (128), the sum and the integral can be interchanged for \( t > 0 \) since the sum is converging absolutely and uniformly. The function \( G_t \) is

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known as the fundamental solution or Green’s function of the heat equation for the $2\pi$ periodic interval. It is immediate from the sum formula (129) that
\[
\int_0^{2\pi} G_t(x) dx = 1.
\] (130)

We now show the following.

**Proposition 13.** Let $G_t(x)$ be as in (129).
\[
G_t(x) = \frac{1}{\sqrt{4\pi t}} \sum_{m=-\infty}^{\infty} \exp(- (x - 2\pi m)^2/(4t)).
\] (131)

**Proof.** Let $Q_t(x)$ be the right hand side of (131) We will show that:
\[
\langle Q_t, e_k \rangle = \frac{1}{\sqrt{2\pi}} \exp(-tk^2).
\] (132)

Once we show this, the result follows since the Fourier series of $Q_t$ is simply the sum in (129). The Fourier sum is equal to $Q_t$ since $Q_t$ is smooth. We thus compute (132).
\[
\langle Q_t, e_k \rangle = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{4\pi t}} \int_0^{2\pi} \sum_{m=-\infty}^{\infty} \exp(- (x - 2\pi m)^2/(4t) - ikx) dx
\]
\[
= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{4\pi t}} \sum_{m=-\infty}^{\infty} \int_0^{2\pi} \exp(- (x - 2\pi m)^2/(4t) - ikx) dx
\]
\[
= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{4\pi t}} \sum_{m=-\infty}^{\infty} \int_{-2\pi m}^{-2\pi (m-1)} \exp(-w^2/(4t) - ik(w + 2\pi m)) dw
\]
\[
= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \exp(-x^2/(4t) - ikx) dx
\] (133)

We thus compute:
\[
\int_{-\infty}^{\infty} \exp(-x^2/(4t) - ikx) dx = \exp(-tk^2) \int_{-\infty}^{\infty} \exp(-(x + 2tk)^2/4t) dx
\]
\[
= \sqrt{4t} \exp(-tk^2) \int_{-\infty}^{\infty} \exp(-(x + \sqrt{t}ki)^2) dx.
\] (134)
We must now compute the last integral. This can be accomplished using contour integration in the complex plane. Let $\sqrt{tk} = a$. We will show that:

$$
\int_{-\infty}^{\infty} \exp(-x^2)dx = \int_{-\infty}^{\infty} \exp(-x^2)dx.
$$

(135)

For simplicity, we let $a > 0$. The case $a < 0$ is exactly the same. Consider the function $\exp(-z^2)$, where $z \in \mathbb{C}$. This function is analytic in the entire complex plane. Consider the following line segments complex plane:

$$
C_{\pm R} = is \pm R, \quad 0 \leq s \leq a,
$$

$$
C_a = ia + s, \quad -R \leq s \leq R,
$$

$$
C_0 = s, \quad -R \leq s \leq R.
$$

(136)

By Cauchy’s theorem, the contour integral composed of the above line segments of the analytic function $\exp(-z^2)$ must be 0:

$$
\int_{C_0} \exp(-z^2)dz + \int_{C_R} \exp(-z^2)dz - \int_{C_a} \exp(-z^2)dz - \int_{C_{-R}} \exp(-z^2)dz = 0
$$

(137)

Now,

$$
\left| \int_{C_{\pm R}} \exp(-z^2)dz \right| \leq a \exp(a^2 - R^2) \to 0 \text{ as } R \to \infty.
$$

(138)

Thus,

$$
\lim_{R \to \infty} \int_{C_0} \exp(-z^2)dz = \lim_{R \to \infty} \int_{C_a} \exp(-z^2)dz.
$$

(139)

This is nothing other than (135). The fact that

$$
\int_{-\infty}^{\infty} \exp(-x^2)dx = \sqrt{\pi}
$$

(140)

is well-known.

What we see from expression (131) is that $G_t(x)$ is positive. This gives us the following important result.

**Proposition 14.** Let $f \in C(\mathbb{T})$. Then,

$$
\|H_t f\|_\infty \leq \|f\|_\infty.
$$

(141)
Proof. We estimate \((H_tf)(x)\).

\[
| (H_tf)(x) | \leq \int_0^{2\pi} |G_t(x-y)| |f(y)| dy = \int_0^{2\pi} G_t(x-y) |f(y)| dy
\]

\[
\leq \int_0^{2\pi} G_t(x-y) dy \|f\|_{\infty} = \|f\|_{\infty},
\]

where we used (128) in the first inequality, the positivity of \(G_t\) in the first equality and (130) in the last equality.

\(\Box\)

One thing that the above shows is that the maximum and minimum values of \((H_tf)(x) = u(x,t)\) are attained at \(t = 0\) (can you see how to prove this?). The above is also crucial for the proof of Theorem 3.

**Proof of Theorem 3.** Let \(f \in C(\mathbb{T})\) and take \(g \in C^2(\mathbb{T})\) such that:

\[
\|f - g\|_{\infty} \leq \epsilon. \tag{143}
\]

Now,

\[
\|H_tf - f\|_{\infty} \leq \|H_t(f-g)\|_{\infty} + \|H_tg - g\|_{\infty} + \|f - g\|_{\infty}
\]

\[
\leq 2\|f - g\|_{\infty} + \|H_tg - g\|_{\infty} \leq 2\epsilon + \|H_tg - g\|_{\infty}, \tag{144}
\]

where we used Proposition 14 in the second inequality. Since \(g \in C^2(\mathbb{T})\), using Proposition 3.2, we obtain the desired result. \(\Box\)