1. Problems from Handout:
   (a) Problem from Section 1.2: 3, 9, 10.
   (b) Problem from Section 1.3: 4.

2. In this problem, we consider a matrix $A$ such that $AA^* = A^*A$, where $A^*$ is the adjoint matrix (conjugate transpose). Such a matrix is called a normal matrix.
   (a) Suppose there are two square matrices $A$ and $B$ that commute: $AB = BA$. Then, show that there is at least one eigenvector that is common to the two matrices. (*Hint: Show any eigenspace of $A$ is invariant under $B$.)
   (b) Let $A$ be square matrix acting on a vector space $V$ and suppose $W \subset V$ is a subspace of $V$. Suppose $W$ is invariant under $A$. Show that $W^\perp$ is also invariant under $A^*$.
   (c) Show that normal matrices can be diagonalized using a unitary matrix.
   (d) Show that any normal matrix $A$ can be written as $A = SU = US$ where $U$ is unitary and $S$ is self-adjoint.

3. Let $A$ be an $n \times n$ real symmetric matrix. Let $H$ be an $n - j$ dimensional subspace of $\mathbb{R}^n$, where $n - j \geq 1$. Let $q_1, \cdots, q_{n-j}$ be a set of orthonormal vectors spanning the subspace $H$. Form the $n \times (n-j)$ matrix with columns $q_1, \cdots, q_{n-j}$ and call this matrix $Q$. Define:
   $$A_H = Q^T A Q.$$ 
   Let the eigenvalues of $A$ be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and let the eigenvalues of $A_H$ be $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_{n-j}$.
   (a) Show that the eigenvalues of $A_H$ do not depend on the choice of orthonormal basis on $H$.
   (b) Let $S_{k-1}$ be the set of all $n - k + 1$ dimensional subspaces of $\mathbb{R}^n$, and $\tilde{S}_{k-1}$ be the set of all $(n - j) - (k - 1)$ dimensional subspaces.
of $H$ (assuming $(n-j)-(k-1) \geq 1$). Show that:

$$\min_{P \in \mathcal{S}_{k-1}} \max_{\|\mathbf{x}\|=1, \mathbf{x} \in P} \langle A\mathbf{x}, \mathbf{x} \rangle \geq \min_{P \in \mathcal{S}_{k-1}} \max_{\|\mathbf{x}\|=1, \mathbf{x} \in P \cap H} \langle A\mathbf{x}, \mathbf{x} \rangle \geq \min_{P \in \mathcal{S}_{k-1}} \max_{\|\mathbf{x}\|=1, \mathbf{x} \in P \cap H} \langle A\mathbf{x}, \mathbf{x} \rangle.$$ 

(c) Use the above to show that:

$$\lambda_k \geq \hat{\lambda}_k \geq \lambda_{k+j}, \quad 1 \leq k \leq n-j.$$ 

This is the content of Theorem 1.8 in handout, with essentially the same proof.

4. Consider the mass-spring problem considered in p. 22-24 of the handout.

(a) Suppose the $n$-th mass is pegged so that $u_n = 0$. Now, we have a $n-1$ mass system rather than an $n$-mass system. What can you say about the frequencies of the new system compared to the original system in which the $n$-th mass was not pegged?

(b) Suppose the spring connecting the $j$-th and $j+1$-th mass is replaced by a massless rigid rod of the same length as the rest of the spring. This means that $u_j = u_{j+1}$. What can you say about the frequencies of the new system compared to the original system?

(c) Suppose you increase the spring constant connecting the $j$-th and $j+1$-th mass to infinity. What do you think (without proof) will happen to the frequencies?