In the sequel, $S_n(f)$ refers to the Fourier partial sum as dealt with in class. For $f$ a function defined on $T = \mathbb{R}/2\pi \mathbb{Z}$,

$$S_n(f) = \sum_{|k| \leq n} \hat{f}(k)e_k, \quad e_k = \frac{1}{\sqrt{2\pi}} \exp(ikx),$$

where

$$\langle f, g \rangle = \int_T fgdx, \quad \hat{f}(k) = \langle f, e_k \rangle.$$

1. Show the following.
   
   (a) For $f \in C(T)$, show that
   
   $$(S_n)^2(f) = S_n(S_n(f)) = S_n(f).$$
   
   (b) For $f, g \in C(T)$, show that
   
   $$\langle S_n(f), g \rangle = \langle f, S_n(g) \rangle.$$
   
   (c) Use the above two results and the Cauchy-Schwartz inequality to show that:

   $$\|S_n(f)\| \leq \|f\|.$$

2. A function $f \in C(T)$ is said to be Hölder continuous of order $\alpha$ if:

$$|f(x) - f(y)| \leq C |x - y|^\alpha, \quad 0 < \alpha < 1, x, y \in T. \quad (1)$$

for some constant $C$ that does not depend on $x$ or $y$. Hölder continuity is a kind of $\alpha$-differentiability. If $f$ is Hölder continuous where $\alpha > 1/2$, show that $S_nf(x)$ converges to $f(x)$ at every point in $T$. (Hint: Use the argument in the proof of Theorem 1.)

3. Suppose $f \in C^1(T)$, and

$$\int_T fdx = 0.$$

Show that

$$\int_T f^2dx \leq \int_T f'^2dx.$$

When does the equality hold? This is an instance of the Poincaré inequality.
4. Consider the following \textit{convolution} operation:

\[ f \ast g \equiv \int_{-\pi}^{\pi} f(y)g(x - y)dy \]

for \( f, g \in C(\mathbb{T}) \).

(a) Show that

\[ \hat{f} \ast \hat{g}(k) = \sqrt{2\pi} \hat{f}(k)\hat{g}(k) \]

(b) Suppose \( f \in C(\mathbb{T}) \) and \( g \in C^\infty(\mathbb{T}) \). Use the above result to argue that \( f \ast g \) should also be \( C^\infty(\mathbb{T}) \).