In the sequel, $S_n(f)$ refers to the Fourier partial sum as dealt with in class. For $f$ a function defined on $T = \mathbb{R}/2\pi \mathbb{Z}$, 

$$S_n(f) = \sum_{|k| \leq n} \hat{f}(k)e_k, \ e_k = \frac{1}{\sqrt{2\pi}} \exp(ikx),$$

where

$$\langle f, g \rangle = \int_T f \bar{g} dx, \ \hat{f}(k) = \langle f, e_k \rangle.$$ 

1. (a) Consider two functions $f, g \in C(T)$. Show that:

$$\langle f, g \rangle = \sum_{k=-\infty}^{\infty} \hat{f}(k)\bar{g}(k). \quad (1)$$

In particular, we have the Parseval identity:

$$\|f\|_2^2 = \langle f, f \rangle = \sum_{k=-\infty}^{\infty} \left| \hat{f}(k) \right|^2. \quad (2)$$

(b) The above identities are true for piecewise continuous functions (in fact, $L^2$ functions). Let

$$f = \begin{cases} \pi - x & \text{if } 0 \leq x < \pi, \\ -\pi - x & \text{if } -\pi \leq x < 0. \end{cases} \quad (3)$$

Apply the Parseval identity to the above function $f$ to compute the sum:

$$\sum_{k=1}^{\infty} \frac{1}{k^2}. \quad (4)$$

2. Consider solving the Laplace equation in a two dimensional disc:

$$\Delta u = 0 \text{ in } |x| < 1, \ u = f \text{ at } |x| = 1. \quad (5)$$

It is convenient to write this in polar coordinates $(r, \theta)$:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \text{ for } r < 1, \ u(r = 1, \theta) = f(\theta). \quad (6)$$
(a) Write the solution $u(r, \theta)$ as:

$$u(r, \theta) = \sum_{k=-\infty}^{\infty} u_k(r) e_k(\theta). \quad (7)$$

Derive the differential equation satisfied by $u_k(r)$.

(b) Check that the general solution to the equations for $u_k(r)$ are given by:

$$u_k(r) = \begin{cases} 
A_k r^{|k|} + B_k r^{-|k|} & \text{if } k \neq 0 \\
A + B \log(r) & \text{if } k = 0.
\end{cases} \quad (8)$$

Using the fact that there should be no singularity at $r = 0$, express $u_k(r)$ and hence $u(r, \theta)$ using the Fourier coefficients of $f$.

(c) From the above expression for $u$, what can you say about the differentiability of $u$ with respect to $r$ and $\theta$?

(d) Derive the Poisson formula:

$$u(r, \theta) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - \eta) + r^2} f(\eta) d\eta. \quad (9)$$

3. Our goal in this problem is to prove the isoperimetric inequality. Consider a closed $C^1$ curve $(x(s), y(s))$ parametrized by arclength so that:

$$x'(s)^2 + y'(s)^2 = 1. \quad (10)$$

Let $L$ be the length of the curve. Since $x(s)$ and $y(s)$ are periodic functions of $L$, we have:

$$x(s) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(2\pi ks/L) + b_k \sin(2\pi ks/L)), \quad (11)$$

$$y(s) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(2\pi ks/L) + d_k \sin(2\pi ks/L)).$$

where $a_k, b_k, c_k, d_k \in \mathbb{R}$.

(a) Compute the length to find that:

$$L = \int_{0}^{L} 1 ds = \int_{0}^{L} (x'(s)^2 + y'(s)^2) ds = \frac{2\pi^2}{L} \sum_{k=1}^{\infty} k^2 (a_k^2 + b_k^2 + c_k^2 + d_k^2). \quad (12)$$
(b) Compute the area to find that:

\[
A = \frac{1}{2} \int_0^L (y'(s)x(s) - x'(s)y(s))ds = \pi \sum_{k=1}^{\infty} k(b_k c_k - a_k d_k). \quad (13)
\]

(c) Show that

\[
L^2 - 4\pi A \geq 0 \quad (14)
\]

and that equality holds only when:

\[
a_1 = -d_1, c_1 = b_1, a_k = b_k = c_k = d_k = 0 \text{ for } k \geq 2. \quad (15)
\]

In this case, the curve is a circle. This is the isoperimetric inequality stating that, for a fixed perimeter \( L \), the shape enclosing the largest area is a circle.

4. Consider:

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L
\]

\[
u(x,0) = f(x)
\]

\[
u = 0 \text{ at } x = 0, \ u + \frac{\partial u}{\partial x} = 0 \text{ at } x = L
\]

(a) Find the eigenvalues \( \lambda_k \) and eigenfunctions \( \phi_k \) satisfying the following:

\[
-\frac{\partial^2 \phi_k}{\partial x^2} = \lambda_k \phi_k,
\]

\[\phi_k = 0 \text{ at } x = 0, \ \phi_k + \frac{\partial \phi_k}{\partial x} = 0 \text{ at } x = L \quad (17)
\]

(b) Show that the \( \phi_k(x) \) are orthogonal to one another in the following sense:

\[
\int_0^L \phi_k \phi_l dx = 0 \text{ if } k \neq l. \quad (18)
\]

(c) Use the above functions to find the solution to equation (16).