Consider the following quasilinear first order equation.

\[ a(x, y, u)u_x + b(x, y, u)u_y + c(x, y, u) = 0. \] (1)

The function \(u(x, y)\) is our unknown, and \(a, b\) and \(c\) are \(C^1\) functions of their arguments. Suppose we are given a function \(u(x, y)\) that satisfies the above equation. Now, consider the curve \(x(t), y(t)\) satisfying

\[ \frac{dx}{dt} = a(x(t), y(t), u(x(t), y(t))), \]
\[ \frac{dy}{dt} = b(x(t), y(t), u(x(t), y(t))). \] (2)

Along this curve, we have:

\[ \frac{d}{dt}u(x(t), y(t)) = u_x \frac{dx}{dt} + u_y \frac{dy}{dt} = au_x + bu_y = -c(x(t), y(t), u(x(t), y(t))). \] (3)

where we used (2) in the second equality and (1) in the last. The curve traced by \((x(t), y(t), u(x(t), y(t)))\) (or \((x(t), y(t)))\) is called a characteristic curve.

The above observation suggests the following method of solving equation (1). Suppose we are given a \(C^1\) curve \(\Gamma\) given by \((x_0(s), y_0(s))\). On this curve, we are given initial values \(u = u_0(s)\). We suppose that the following condition is satisfied on this curve:

\[ \text{det} \begin{pmatrix} \frac{dx_0}{ds} & a \\ \frac{dy_0}{ds} & b \end{pmatrix} \neq 0. \] (4)
This is called the noncharacteristic condition. Now, for each value of \( s \), we solve the following differential equation:

\[
\frac{dx}{dt} = a(x, y, u), \quad \frac{dy}{dt} = b(x, y, u), \quad \frac{du}{dt} = -c(x, y, u) \tag{5}
\]

with initial condition:

\[
x(t = 0, s) = x_0(s), \quad y(t = 0, s) = y_0(s), \quad u(t = 0, s) = u_0(s). \tag{6}
\]

We have now expressed \( x, y \) and \( u \) as a function of \( (t, s) \). Given the non-characteristic condition (4), we can solve for \( (t, s) \) in terms of \( (x, y) \) (by the inverse function theorem) near the curve \( \Gamma \). We may then substitute this into \( u(t, s) \) to find the function \( u(x, y) \).

We can check that this indeed gives us the solution in the following way. Note first that \( u(x, y) \) is a \( C^1 \) function (by \( C^1 \) dependence of solutions of ODEs to parameters and the inverse function theorem). Thus, using (5)

\[
\frac{\partial}{\partial t}(u(x(t, s), y(t, s))) = u_x \frac{\partial x}{\partial t} + u_y \frac{\partial y}{\partial t} = au_x + bu_y. \tag{7}
\]

On the other hand, again using (5),

\[
\frac{\partial}{\partial t}(u(x(t, s), y(t, s))) = -c, \tag{8}
\]

and therefore, (1) is satisfied.

We would now like to generalize this construction to the fully nonlinear situation. Consider:

\[
F(x, y, u, p, q) = 0, \quad p = u_x, \quad q = u_y. \tag{9}
\]

We assume \( F \) is a \( C^1 \) function of its arguments. Let the \( C^2 \) function \( u(x, y) \) solve this PDE. We want to find a characteristic ODE system similar to (5).

To do so, we first set:

\[
\frac{dx}{dt} = \frac{\partial F}{\partial p} = F_p, \quad \frac{dy}{dt} = \frac{\partial F}{\partial q} = F_q. \tag{10}
\]

This is in analogy with (5). Indeed, applying this procedure to (1) results in (5). Given this, we may find the equations for \( u \) along the characteristic lines as follows.

\[
\frac{d}{dt} u(x(t), y(t)) = u_x \frac{dx}{dt} + u_y \frac{dy}{dt} = pF_p + qF_q \tag{11}
\]
We would now like to use (10) with (11) as our characteristic ODE system. However, this is not possible since the ODE system, unlike (5) depends not only on $x, y$ and $u$ but also on $u_x$ and $u_y$. Therefore, we must derive equations for $p = u_x$ and $q = u_y$. We have:

$$\frac{dp}{dt} = \frac{d}{dt}u_x = u_{xx} \frac{dx}{dt} + u_{xy} \frac{dy}{dt} = u_{xx}F_p + u_{xy}F_q \tag{12}$$

Let us take the derivative of (9) with respect to $x$:

$$F_x + F_u u_x + F_p u_{xx} + F_q u_{xy} = 0. \tag{13}$$

Therefore, (12) may be written as

$$\frac{dp}{dt} = -F_x - pF_u. \tag{14}$$

Likewise, we have:

$$\frac{dq}{dt} = -F_y - qF_u. \tag{15}$$

We now have a closed ODE system, which we list here below for convenience:

$$\frac{dx}{dt} = F_p, \quad \frac{dy}{dt} = F_q, \quad \frac{du}{dt} = pF_p + qF_q, \quad \frac{dp}{dt} = -pF_u - F_x, \quad \frac{dq}{dt} = -qF_u - F_y. \tag{16}$$

The above is called the Lagrange-Charpit system of ODEs.

This leads to the following method for solving (9). First, we are given a non-characteristic curve $\Gamma$ given by $(x_0(s), y_0(s))$ and values $u = u_0(s)$ on this curve. In contrast to the quasilinear case (1), we need initial conditions for $p = p_0(s)$ and $q_0(s)$ to solve (16). The initial conditions must satisfy the PDE (9):

$$F(x_0, y_0, u_0, p_0, q_0) = 0. \tag{17}$$

We need another condition to determine $p_0$ and $q_0$. To obtain this condition, note that the solution $u(x, y)$ to (9) must satisfy

$$\frac{d}{ds}u(x_0(s), y_0(s)) = u_x \frac{dx_0}{ds} + u_y \frac{dy_0}{ds}. \tag{18}$$

Therefore, we require that $p_0$ and $q_0$ also satisfy:

$$\frac{du_0}{ds} = p_0(s) \frac{dx_0}{ds} + q_0(s) \frac{dy_0}{ds}. \tag{19}$$
Equations (17) and (19) may be solved for each $s$ to obtain the initial functions $p_0(s)$ and $q_0(s)$. One difference between (9) and (1) is that there could be multiple solutions $p_0$ and $q_0$. Once the functions $p_0$ and $q_0$ are chosen, one requires that the following non-characteristic condition is satisfied:

$$\det \begin{pmatrix} x_0'(s) & F_p \\ y_0'(s) & F_q \end{pmatrix} \neq 0. \quad (20)$$

We may now solve (16) for each $s$ with initial conditions given on the non-characteristic curve. This procedure produces the functions:

$$x(t, s), \ y(t, s), \ u(t, s), \ p(t, s), \ q(t, s). \quad (21)$$

We may solve for $t$ and $s$ in terms of $x$ and $y$ (thanks to (20) and the inverse function theorem), and substitute this into the expression $u(t, s)$ to obtain $u$ as a function $x$ and $y$. We thus obtain a solution $u(x, y)$ in a neighborhood $\mathcal{U}$ of $\Gamma$.

We now show that this procedure indeed produces an equation that satisfies (9). We first show that

$$G(t, s) \equiv F(x(t, s), y(t, s), u(t, s), p(t, s), q(t, s)) = 0. \quad (22)$$

Note first that $G(0, s) = 0$ by design (see (17)). We see that

$$\frac{\partial G}{\partial t} = F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_u \frac{du}{dt} + F_p \frac{dp}{dt} + F_q \frac{dq}{dt} = F_x F_p + F_y F_q + F_u (pF_p + qF_q) - F_p (F_x + pF_u) - F_q (F_y + qF_u) \quad (23)$$

Thus, $G(t, s) = 0$. Next, we must check that:

$$p(x, y) = \frac{\partial u}{\partial x}(x, y), \ q(x, y) = \frac{\partial u}{\partial y}(x, y). \quad (24)$$

To check this, we show that the function

$$H(t, s) \equiv \frac{\partial u}{\partial s} - p(t, s) \frac{\partial x}{\partial s} - q(t, s) \frac{\partial y}{\partial s} = 0. \quad (25)$$

Note that $H(0, s) = 0$ given (19). Let us compute the derivative of $H$ with
respect to $t$.

$$
\frac{\partial H}{\partial t} = \frac{\partial^2 u}{\partial t \partial s} - \frac{\partial p}{\partial t} \frac{\partial x}{\partial s} - p \frac{\partial^2 x}{\partial t \partial s} - \frac{\partial q}{\partial t} \frac{\partial y}{\partial s} - q \frac{\partial^2 y}{\partial t \partial s}
$$

$$
= \frac{\partial}{\partial s} (pF_p + qF_q) + (F_x + pF_u) \frac{\partial x}{\partial s} - p \frac{\partial}{\partial s} F_p + (F_y + qF_u) \frac{\partial y}{\partial s} - q \frac{\partial}{\partial s} F_q
$$

$$
= F_p \frac{\partial p}{\partial s} + F_q \frac{\partial q}{\partial s} + (F_x + pF_u) \frac{\partial x}{\partial s} + (F_y + qF_u) \frac{\partial y}{\partial s}
$$

(26)

Recall that $G(t, s) = 0$ for all $(t, s)$. If we take the derivative of this with respect to $s$, we obtain:

$$
\frac{\partial G}{\partial s} = F_x \frac{\partial x}{\partial s} + F_y \frac{\partial y}{\partial s} + F_u \frac{\partial u}{\partial s} + F_p \frac{\partial p}{\partial s} + F_q \frac{\partial q}{\partial s} = 0.
$$

(27)

Using this fact in the last line of (26), we obtain:

$$
\frac{\partial H}{\partial t} = -F_u \left( \frac{\partial u}{\partial s} - \frac{\partial x}{\partial s} - \frac{\partial y}{\partial s} \right) = -F_u H.
$$

(28)

This is an ODE for each value of $s$. The solution to the above ODE with initial condition $H(0, s) = 0$ is $H(t, s) = 0$. We now show that $H(t, s) = 0$ implies (24). To see this, note that

$$
\frac{\partial u}{\partial t} = pF_p + qF_q = p \frac{\partial x}{\partial t} + q \frac{\partial y}{\partial t}.
$$

(29)

Thus, the above together with (25) shows that

$$
\begin{pmatrix}
\frac{\partial u}{\partial t} \\
\frac{\partial u}{\partial s}
\end{pmatrix}
= J
\begin{pmatrix}
p \\
q
\end{pmatrix},
J
= \begin{pmatrix}
\frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \\
\frac{\partial x}{\partial s} & \frac{\partial y}{\partial s}
\end{pmatrix}
$$

(30)

Viewing $u$ as a function of $x$ and $y$, we have

$$
J
\begin{pmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial u}{\partial t} \\
\frac{\partial u}{\partial s}
\end{pmatrix}
$$

(31)

Using (30) and (31) and the fact that $J$ is invertible (thanks to the inverse function theorem and (20)) we obtain the equality (24).