1. Consider the following equation:

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( \frac{1}{\epsilon} b \left( \frac{x}{\epsilon} \right) u + D \frac{\partial u}{\partial x} \right)
\]

(1)

where \( D \) is a positive constant and \( b(y) \) is a periodic function in \( y \) with period 1 whose average is equal to 0:

\[
\int_0^1 b(y)dy = 0.
\]

(2)

We will be interested in positive solutions \((u > 0)\). Assume that \( u(x,t) \) can be written as:

\[
u(x,t) = u_0(x,y,t) + \epsilon u_1(x,y,t) + \epsilon^2 u_2(x,y,t) + \cdots, \quad y = x/\epsilon.
\]

(3)

(a) Show that \( u_0(x,y,t) \) can be written as:

\[
u_0(x,y,t) = \hat{u}(x,t)v_0(y)
\]

(4)

where \( v_0(y) \) is a function that does not depend on \( x \) or \( t \).

(b) Show that \( \hat{u}(x,t) \) satisfies the equation:

\[
\frac{\partial \hat{u}}{\partial t} = \hat{D} \frac{\partial^2 \hat{u}}{\partial x^2}.
\]

Obtain an explicit expression for \( \hat{D} \) and show that \( \hat{D} \) is never greater than \( D \).

2. Consider the system:

\[
\epsilon \frac{\partial u}{\partial t} = \epsilon^2 \frac{\partial^2 u}{\partial x^2} + f(u,v), \quad f(u,v) = u(1-u)(u-1-v),
\]

(6)

\[
\epsilon \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - f(u,v).
\]

(7)

posed on \( 0 < x < 1 \) with boundary conditions:

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \text{ at } x = 0, 1.
\]

(8)

Values of \( u \) and \( v \) are assumed positive.
(a) Show that
\[ \int_0^1 (u + v) dx = K \] (9)
where \( K \) is a constant independent of time.

(b) Show that, to leading order, \( v \) is constant in \( x \) (but may vary in time) and \( u \) can only take values 0 or \( 1 + v \).

(c) Given the above result, there must be one or more sharp transition layers within the domain, across which \( u \) changes from 0 to \( 1 + v \). Let there be one such front, and let \( \phi(t) \) be the position of this front; to leading order suppose
\[ u = \begin{cases} 
1 + v & \text{if } x < \phi(t), \\
0 & \text{if } x > \phi(t). 
\end{cases} \] (10)

Find an ordinary differential equation satisfied by this front position in terms of \( \phi \) and \( K \) only.

In solving the above problem, it may be useful to use the following fact. Suppose we are given the equation:
\[ -c \frac{dw}{d\xi} = \frac{\partial^2 w}{\partial \xi^2} + w(1 - w)(w - b), \quad b > 1, \]
\[ \lim_{\xi \to -\infty} w(\xi) = b, \quad \lim_{\xi \to \infty} w(\xi) = 0. \] (11)

Then, there is a solution \( w \) only when:
\[ c = \frac{1}{\sqrt{2}}(b - 2) \] (12)
and \( w \) is given by:
\[ w = \frac{b}{2} \left( 1 - \tanh \left( \frac{b\xi}{\sqrt{2}} + k \right) \right) \] (13)
where \( k \) is arbitrary.

3. Consider the following Cahn-Hilliard equation in \( \Omega \subset \mathbb{R}^2 \)
\[ \frac{\partial u}{\partial t} = -\Delta \left( \epsilon \Delta u + \frac{1}{\epsilon} f(u) \right), \] (14)
where \( f(u) \) is the usual cubic nonlinearity as in the Allen-Cahn equation. The stable roots are at \( u = 0 \) and \( u = 1 \), and we consider the balanced case:

\[
\int_0^1 f(s) ds = 0. \tag{15}
\]

Rewrite this equation as:

\[
\frac{\partial u}{\partial t} = \Delta w \tag{16}
\]

\[-w = \epsilon \Delta u + \frac{1}{\epsilon} f(u). \tag{17}\]

The boundary condition imposed are:

\[
\frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0 \text{ on } \partial \Omega. \tag{18}
\]

Thus, no-flux boundary conditions are imposed on both \( u \) and \( w \).

(a) Expand \( u \) and \( w \) as:

\[
u = u_0 + \epsilon u_1 + \cdots, \quad w = w_0 + \epsilon w_1 \tag{19}\]

and find the equations satisfied \( u_0 \) and \( w_0 \). You should find that there will be regions where either \( u_0 = 0 \) or \( u_0 = 1 \).

(b) Let \( \Gamma \) be the interface separating the region where \( u_0 = 0 \) and \( u_0 = 1 \). Introduce a moving coordinate system that fits with \( \Gamma \) and find the equations in the inner transition layers together with matching conditions.

(c) Show that \( w_0 \) satisfies the following Mullins Sekerka problem:

\[
\Delta w_0 = 0 \text{ for } \Omega \setminus \Gamma \tag{20}
\]

\[
w_0 = -A \kappa, \quad v = \left[ \frac{\partial w_0}{\partial \nu} \right] \text{ on } \Gamma \tag{21}
\]

where \( A \) is a positive constant that only depends on the cubic nonlinearity \( f \), \( \kappa \) is the curvature of the curve \( \Gamma \) and \( \nu \) is the unit normal vector on \( \Gamma \), \( v \) is the normal velocity of the curve \( \Gamma \), and \([\cdot]\) denotes the jump in the enclosed quantity across the interface \( \Gamma \).

4. Consider a fluid placed between two plates that are a distance \( \epsilon L \) apart. In terms of coordinates, the two plates correspond to \( z = 0 \) and
\( z = \epsilon L(x, y) \), where \( L(x, y) \) is some function of \( x \) and \( y \). The equation satisfied by the fluid is:

\[
\mu \Delta u = \nabla p, \quad \nabla \cdot u = 0 \quad \text{for } 0 < z < \epsilon L \quad (22)
\]

\[
u = 0 \quad \text{on } z = 0, \epsilon L. \quad (23)
\]

We consider the limit when \( \epsilon \) is small. Such a flow is called Hele-Shaw flow.

(a) Let \( u = (u, v, w)^T \). Rescale the equations so that \( z = \epsilon Z \).

(b) Expand \( u, v, w, p \) in powers of epsilon (let \( u = u_0 + \epsilon u_1 + \epsilon^2 u_2 \) and so on). Show that \( u_0, u_1, v_0, v_1, w_0, w_1, w_2 \) are all 0. Show that \( p_0 \) does not depend on \( Z \). (This shows that the leading order contribution to the velocity is order \( \epsilon^2 \)).

(c) Express \( u_2, v_2 \) in terms of \( p_0 \).

(d) Find the equation satisfied by \( p_0 \) (Hint: This comes from the incompressibility condition \( \nabla \cdot u = 0 \)).