Consider the following eigenvalue problem:

\[-\frac{d^2u}{dx^2} + \epsilon f(x)u = \lambda u, \quad \text{in } \mathbb{R}/2\pi\mathbb{Z},\]  

where $f$ is a real smooth periodic function. Here, $\lambda$ is an eigenvalue and $u$ is the eigenfunction.

1. Find all eigenvalues and eigenfunctions of the above problem when $\epsilon = 0$.
2. Discuss what happens with the eigenvalues and eigenfunctions when $\epsilon \neq 0$ but small.

Answers:

First, let us consider the eigenvectors for $\epsilon = 0$. I will just write out the solution:

\[
\lambda = 0, \quad u = e_0 = \frac{1}{\sqrt{2\pi}}; \quad u = e_n = \frac{1}{\sqrt{2\pi}} \exp(inx) \quad \text{and} \quad u = e_{-n} = \frac{1}{\sqrt{2\pi}} \exp(-inx),
\]

where $N$ is the set of positive integers. There is no need to place the $1/\sqrt{2\pi}$ factor in front, but this is convenient because, then, $u_n$ and $v_n$ will have unit length under the standard $L^2$ inner product:

\[
\langle f, g \rangle = \int_{0}^{2\pi} fgdx.
\]

The eigenvectors for $\lambda = n^2$ can also be written as:

\[
u = e_n = \frac{1}{\sqrt{\pi}} \cos(nx), \quad u = s_n = \frac{1}{\sqrt{\pi}} \sin(nx).
\]

Now, let us consider what happens to the eigenvalues/eigenvectors when $\epsilon \neq 0$. Expand $\lambda$ as:

\[
\lambda = \lambda_0 + \epsilon \lambda_1 + \cdots, \quad u = u_0 + \epsilon u_1 + \cdots.
\]
The leading order equation is:

\[- \frac{d^2 u_0}{dx^2} = \lambda_0 u_0.\]  \hspace{1cm} (7)

This is the unperturbed problem, that we just solved. We must consider two cases, \( \lambda_0 = 0 \) and \( \lambda_0 = n^2 \). When \( \lambda_0 = 0 \), \( u_0 = e_0 \), and we have:

\[- \frac{d^2 u_1}{dx^2} = \lambda_1 e_0 - fe_0.\]  \hspace{1cm} (8)

In order for \( u_1 \) to have a solution, the right hand side must be orthogonal to the constant function by the Fredholm Alternative. Thus,

\[\langle \lambda_1 e_0, e_0 \rangle = \langle fe_0, e_0 \rangle.\]  \hspace{1cm} (9)

Thus,

\[\lambda_1 = \frac{\langle fe_0, e_0 \rangle}{\langle e_0, e_0 \rangle} = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx.\]  \hspace{1cm} (10)

For \( \lambda = n^2 \), we may proceed similarly to find:

\[- \frac{d^2 u_1}{dx^2} + n^2 u_1 = \lambda_1 u_0 - f u_0.\]  \hspace{1cm} (11)

By the Fredholm alternative, the right hand side must be perpendicular to both \( e_n \) and \( e_{-n} \). Thus,

\[\langle \lambda_1 u_0 - f u_0, e_n \rangle = 0 \quad \text{and} \quad \langle \lambda_1 u_0 - f u_0, e_{-n} \rangle = 0.\]  \hspace{1cm} (12)

Now, \( u_0 \) is spanned by \( e_n \) and \( e_{-n} \). Therefore,

\[u_0 = ae_n + be_{-n}\]  \hspace{1cm} (13)

for some constants \( a \) and \( b \). Plugging this into (12), we obtain the equation:

\[
\begin{pmatrix}
\langle fe_n, e_n \rangle & \langle fe_{-n}, e_n \rangle \\
\langle fe_n, e_{-n} \rangle & \langle fe_{-n}, e_{-n} \rangle
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
= \lambda_1
\begin{pmatrix}
a \\
b
\end{pmatrix}
\]  \hspace{1cm} (14)

Solving this \( 2 \times 2 \) eigenvalue problem will yield the leading order correction \( \lambda_1 \), and the corresponding eigenvector \( (a, b)^T \) together with (13) will give the eigenvector to leading order.

You can, of course, also use (5) as your eigenfunctions, in which case you will express \( u_0 \) as:

\[u_0 = ac_n + bs_n.\]  \hspace{1cm} (15)
With this choice, the $2 \times 2$ eigenvalue problem to be solved will be:

$$\begin{pmatrix} \langle f_{c_n}, c_n \rangle & \langle f_{s_n}, c_n \rangle \\ \langle f_{c_n}, s_n \rangle & \langle f_{s_n}, s_n \rangle \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

(16)

It is easily seen that (14) or (16) will yield the same answer, because the two matrices are connected via a change of bases.