Consider the following quasilinear first order equation.

\[ a(x, y, u)u_x + b(x, y, u)u_y + c(x, y, u) = 0. \] (1)

The function \( u(x, y) \) is our unknown, and \( a, b \) and \( c \) are \( C^1 \) functions of their arguments. Suppose we are given a function \( u(x, y) \) that satisfies the above equation. Now, consider the curve \( x(t), y(t) \) satisfying

\[
\begin{align*}
\frac{dx}{dt} &= a(x(t), y(t), u(x(t), y(t))), \\
\frac{dy}{dt} &= b(x(t), y(t), u(x(t), y(t))).
\end{align*}
\] (2)

Along this curve, we have:

\[
\frac{d}{dt}u(x(t), y(t)) = u_x \frac{dx}{dt} + u_y \frac{dy}{dt} = au_x + bu_y = -c(x(t), y(t), u(x(t), y(t))).
\] (3)

where we used (2) in the second equality and (1) in the last. The curve traced by \( (x(t), y(t), u(x(t), y(t)) \) (or \( (x(t), y(t)) \)) is called a characteristic curve.

The above observation suggests the following method of solving equation (1). Suppose we are given a \( C^1 \) curve \( \Gamma \) given by \( (x_0(s), y_0(s)) \). On this curve, we are given initial values \( u = u_0(s) \). We suppose that the following condition is satisfied on this curve:

\[
\det \begin{pmatrix} \frac{dx_0}{ds} & a \\ \frac{dy_0}{ds} & b \end{pmatrix} \neq 0.
\] (4)
This is called the noncharacteristic condition. Now, for each value of $s$, we solve the following differential equation:

$$\frac{dx}{dt} = a(x, y, u), \quad \frac{dy}{dt} = b(x, y, u), \quad \frac{du}{dt} = -c(x, y, u) \quad (5)$$

with initial condition:

$$x(t = 0, s) = x_0(s), \quad y(t = 0, s) = y_0(s), \quad u(t = 0, s) = u_0(s). \quad (6)$$

We have now expressed $x, y$ and $u$ as a function of $(t, s)$. Given the noncharacteristic condition (4), we can solve for $(t, s)$ in terms of $(x, y)$ (by the inverse function theorem) near the curve $\Gamma$. We may then substitute this into $u(t, s)$ to find the function $u(x, y)$.

We can check that this indeed gives us the solution in the following way. Note first that $u(x, y)$ is a $C^1$ function (by $C^1$ dependence of solutions of ODEs to parameters and the inverse function theorem). Thus, using (5)

$$\frac{\partial}{\partial t}(u(x(t, s), y(t, s))) = u_x \frac{\partial x}{\partial t} + u_y \frac{\partial y}{\partial t} = au_x + bu_y. \quad (7)$$

On the other hand, again using (5),

$$\frac{\partial}{\partial t}(u(x(t, s), y(t, s))) = -c, \quad (8)$$

and therefore, (1) is satisfied.

We would now like to generalize this construction to the fully nonlinear situation. Consider:

$$F(x, y, u, p, q) = 0, \quad p = u_x, \quad q = u_y. \quad (9)$$

We assume $F$ is a $C^1$ function of its arguments. Let the $C^2$ function $u(x, y)$ solve this PDE. We want to find a characteristic ODE system similar to (5). To do so, we first set:

$$\frac{dx}{dt} = \frac{\partial F}{\partial p} = F_p, \quad \frac{dy}{dt} = \frac{\partial F}{\partial q} = F_q. \quad (10)$$

This is in analogy with (5). Indeed, applying this procedure to (1) results in (5). Given this, we may find the equations for $u$ along the characteristic lines as follows.

$$\frac{d}{dt} u(x(t), y(t)) = u_x \frac{dx}{dt} + u_y \frac{dy}{dt} = pF_p + qF_q \quad (11)$$
We would now like to use (10) with (11) as as our characteristic ODE system. However, this is not possible since the ODE system, unlike (5) depends not only on $x, y$ and $u$ but also on $u_x$ and $u_y$. Therefore, we must derive equations for $p = u_x$ and $q = u_y$. We have:

$$\frac{dp}{dt} = \frac{d}{dt} u_x = u_{xx} \frac{dx}{dt} + u_{xy} \frac{dy}{dt} = u_{xx} F_p + u_{xy} F_q$$  \hspace{1cm} (12)$$

Let us take the derivative of (9) with respect to $x$:

$$F_x + F_u u_x + F_p u_{xx} + F_q u_{xy} = 0.$$  \hspace{1cm} (13)$$

Therefore, (12) may be written as

$$\frac{dp}{dt} = -F_x - p F_u.$$  \hspace{1cm} (14)$$

Likewise, we have:

$$\frac{dq}{dt} = -F_y - q F_u.$$  \hspace{1cm} (15)$$

We now have a closed ODE system, which we list here below for convenience:

$$\frac{dx}{dt} = F_p, \quad \frac{dy}{dt} = F_q, \quad \frac{du}{dt} = p F_p + q F_q,$$

$$\frac{dp}{dt} = -p F_u - F_x, \quad \frac{dq}{dt} = -q F_u - F_y.$$  \hspace{1cm} (16)$$

The above is called the Lagrange-Charpit system of ODEs.

This leads to the following method for solving (9). First, we are given a non-characteristic curve $\Gamma$ given by $(x_0(s), y_0(s))$ and values $u = u_0(s)$ on this curve. In contrast to the quasilinear case (1), we need initial conditions for $p = p_0(s)$ and $q_0(s)$ to solve (16). The initial conditions must satisfy the PDE (9):

$$F(x_0, y_0, u_0, p_0, q_0) = 0.$$  \hspace{1cm} (17)$$

We need another condition to determine $p_0$ and $q_0$. To obtain this condition, note that the solution $u(x, y)$ to (9) must satisfy

$$\frac{d}{ds} u(x_0(s), y_0(s)) = u_x \frac{dx_0}{ds} + u_y \frac{dy_0}{ds}.$$  \hspace{1cm} (18)$$

Therefore, we require that $p_0$ and $q_0$ also satisfy:

$$\frac{du_0}{ds} = p_0(s) \frac{dx_0}{ds} + q_0(s) \frac{dy_0}{ds}.$$  \hspace{1cm} (19)$$
Equations (17) and (19) may be solved for each \( s \) to obtain the initial functions \( p_0(s) \) and \( q_0(s) \). One difference between (9) and (1) is that there could be multiple solutions \( p_0 \) and \( q_0 \). Once the functions \( p_0 \) and \( q_0 \) are chosen, one requires that the following non-characteristic condition is satisfied:

\[
\det \begin{pmatrix} x'(s) & F_p \\ y'(s) & F_q \end{pmatrix} \neq 0. 
\] (20)

We may now solve (16) for each \( s \) with initial conditions given on the non-characteristic curve. This procedure produces the functions:

\[
x(t, s), \ y(t, s), \ u(t, s), \ p(t, s), \ q(t, s).
\] (21)

We may solve for \( t \) and \( s \) in terms of \( x \) and \( y \) (thanks to (20) and the inverse function theorem), and substitute this into the expression \( u(t, s) \) to obtain \( u \) as a function \( x \) and \( y \). We thus obtain a solution \( u(x, y) \) in a neighborhood \( \mathcal{U} \) of \( \Gamma \).

We now show that this procedure indeed produces an equation that satisfies (9). We first show that

\[
G(t, s) \equiv F(x(t, s), y(t, s), u(t, s), p(t, s), q(t, s)) = 0.
\] (22)

Note first that \( G(0, s) = 0 \) by design (see (17)). We see that

\[
\frac{\partial G}{\partial t} = F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_u \frac{du}{dt} + F_p \frac{dp}{dt} + F_q \frac{dq}{dt}
\]

\[
= F_x F_p + F_y F_q + F_u (pF_p + qF_q) - F_p(F_x + pF_u) - F_q(F_y + qF_u) \quad (23)
\]

\[
= 0.
\]

Therefore, \( G(t, s) = 0 \). Next, we must check that:

\[
p(x, y) = \frac{\partial u}{\partial x}(x, y), \ q(x, y) = \frac{\partial u}{\partial y}(x, y).
\] (24)

To check this, we show that the function

\[
H(t, s) \equiv \frac{\partial u}{\partial s} - p(t, s) \frac{\partial x}{\partial s} - q(t, s) \frac{\partial y}{\partial s} = 0.
\] (25)

Note that \( H(0, s) = 0 \) given (19). Let us compute the derivative of \( H \) with
respect to $t$.

\[
\frac{\partial H}{\partial t} = \frac{\partial^2 u}{\partial t \partial s} - \frac{\partial p \partial x}{\partial t \partial s} - p \frac{\partial x}{\partial t} \frac{\partial p}{\partial s} - q \frac{\partial y}{\partial t} \frac{\partial q}{\partial s} - q \frac{\partial^2 y}{\partial t \partial s} =
\]

\[
\frac{\partial}{\partial s} (pF_p + qF_q) + (F_x + pF_u) \frac{\partial x}{\partial s} - p \frac{\partial}{\partial s} F_p + (F_y + qF_u) \frac{\partial y}{\partial s} - q \frac{\partial}{\partial s} F_q
\]

\[
= F_p \frac{\partial p}{\partial s} + F_q \frac{\partial q}{\partial s} + (F_x + pF_u) \frac{\partial x}{\partial s} + (F_y + qF_u) \frac{\partial y}{\partial s}
\]

(26)

Recall that $G(t, s) = 0$ for all $(t, s)$. If we take the derivative of this with respect to $s$, we obtain:

\[
\frac{\partial G}{\partial s} = F_x \frac{\partial x}{\partial s} + F_y \frac{\partial y}{\partial s} + F_u \frac{\partial u}{\partial s} + F_p \frac{\partial p}{\partial s} + F_q \frac{\partial q}{\partial s} = 0.
\]

(27)

Using this fact in the last line of (26), we obtain:

\[
\frac{\partial H}{\partial t} = -F_u \left( \frac{\partial u}{\partial s} - p \frac{\partial x}{\partial s} - q \frac{\partial y}{\partial s} \right) = -F_u H.
\]

(28)

This is an ODE for each value of $s$. The solution to the above ODE with initial condition $H(0, s) = 0$ is $H(t, s) = 0$. We now show that $H(t, s) = 0$ implies (24). To see this, note that

\[
\frac{\partial u}{\partial t} = pF_p + qF_q = p \frac{\partial x}{\partial t} + q \frac{\partial y}{\partial t}.
\]

(29)

Thus, the above together with (25) shows that

\[
\begin{pmatrix} \partial u / \partial t \\ \partial u / \partial s \end{pmatrix} = J \begin{pmatrix} p \\ q \end{pmatrix}, \quad J = \begin{pmatrix} \partial x / \partial t & \partial y / \partial t \\ \partial x / \partial s & \partial y / \partial s \end{pmatrix}
\]

(30)

Viewing $u$ as a function of $x$ and $y$, we have

\[
J \begin{pmatrix} \partial u / \partial x \\ \partial u / \partial y \end{pmatrix} = \begin{pmatrix} \partial u / \partial t \\ \partial u / \partial s \end{pmatrix}
\]

(31)

Using (30) and (31) and the fact that $J$ is invertible (thanks to the inverse function theorem and (20)) we obtain the equality (24).