# SOLUTIONS TO ALGEBRAIC GEOMETRY AND ARITHMETIC CURVES BY QING LIU

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I will collect my solutions to some of the exercises in this book in this document.

## Section 2.1

**1.** Let $A = k[[T]]$ be the ring of formal power series with coefficients in a field $k$. Determine $\operatorname{Spec} A$.

Note that every nonzero $f \in A$ can be written as $f = T^n g$ where $n \geq 0$ and $g$ is a power series with nonzero constant term, in other words a unit in A. Thus if $\mathfrak{p}$ is a nonzero prime ideal in $A$, then picking $0 \neq f \in \mathfrak{p}$ with $f = T^n g$ as above, primeness forces $T^n \in \mathfrak{p}$. Here $n$ must be greater than zero, so using the primeness again we conclude $T \in \mathfrak{p}$. Thus $(T) \subseteq \mathfrak{p}$, but $(T)$ is a maximal ideal so $\mathfrak{p} = (T)$. Therefore

$$\operatorname{Spec} A = \{0, (T)\}$$

whose closed subsets are $\emptyset$, $\operatorname{Spec} A$ and $\{(T)\}$.

**2.** Let $\varphi : A \to B$ be a homomorphism of finitely generated algebras over a field. Show that the image of a closed point under $\operatorname{Spec} \varphi$ is a closed point.

Write $k$ for the underlying field. Let's parse the statement. A closed point in $\operatorname{Spec} B$ means a maximal ideal $\mathfrak{n}$ of $B$. And $\operatorname{Spec}(\varphi)(\mathfrak{n}) = \varphi^{-1}(\mathfrak{n})$. So we want to show that $\mathfrak{p} := \varphi^{-1}(\mathfrak{n})$ is a maximal ideal in $A$. First of all, $\mathfrak{p}$ is definitely a prime ideal of $A$ and $\varphi$ descends to an injective $k$-algebra homomorphism

$$\psi : A/\mathfrak{p} \to B/\mathfrak{n}\,.$$

But the map $k \to B/\mathfrak{n}$ defines a finite field extension of $k$ by Corollary 1.12. So the integral domain $A/\mathfrak{p}$ is trapped between a finite field extension. Such domains are necessarily fields, thus $\mathfrak{p}$ is maximal in $A$.

**3.** Let $k = \mathbb{R}$ be the field of real numbers. Let $A = k[X, Y]/(X^2 + Y^2 + 1)$. We wish to describe $\operatorname{Spec} A$. Let $x, y$ be the respective images of $X, Y$ in $A$.

(a) Let $\mathfrak{m}$ be a maximal ideal of $A$. Show that there exist $a, b, c, d \in k$ such that $x^2 + ax + b$, $y^2 + cy + d \in \mathfrak{m}$. Using the relation $x^2 + y^2 + 1 = 0$, show that $\mathfrak{m}$ contains an element $f = \alpha x + \beta y + \gamma$ with $(\alpha, \beta) \neq (0, 0)$. Deduce from this that $\mathfrak{m} = fA$.

Let's first show the last claim. So assume that $f = \alpha x + \beta y + \gamma$ such that $(\alpha, \beta) \neq (0, 0)$ lies in $\mathfrak{m}$. WLOG we may assume $\beta \neq 0$ and then by scaling $f$ we may assume $\beta = 1$.

Then

$$\begin{aligned} A/fA &\cong \mathbb{R}[X,Y]/(X^2+Y^2+1, Y+\alpha X+\gamma) \\ &\cong \mathbb{R}[X]/(X^2+(-\alpha X-\gamma)^2+1) \\ &= \mathbb{R}[X]/((\alpha^2+1)X^2-2\alpha\gamma X+\gamma^2+1) \end{aligned}$$

Note that the discriminant of the quadratic polynomial we are factoring out from $\mathbb{R}[X]$ at the last step is

$$4\alpha^2\gamma^2 - 4(\alpha^2+1)(\gamma^2+1) = -4(\gamma^2+\alpha^2+1) < 0$$

therefore the quadratic is irreducible hence it generates a maximal ideal in $\mathbb{R}[X]$ and the corresponding quotient is a field. Thus $A/fA$ is a field and hence $fA$ is a maximal ideal of $A$. But we have $fA \subseteq \mathfrak{m}$, therefore $fA = \mathfrak{m}$.

To show such an $f$ exists, first note that by Corollary 1.12 $A/\mathfrak{m}$ is a finite extension of $\mathbb{R}$, hence is isomorphic to $\mathbb{R}$ or $\mathbb{C}$, hence $[A/\mathfrak{m} : \mathbb{R}] \leq 2$. If $\overline{x} \in \mathbb{R}$, then $x+\gamma \in \mathfrak{m}$ for some $\gamma \in \mathbb{R}$ and if $\overline{y} \in \mathbb{R}$ then $y+\gamma \in \mathfrak{m}$ for some $\gamma \in \mathbb{R}$. So we can take $f$ to be this linear polynomial.

Assume neither $\overline{x}$ nor $\overline{y}$ lie in $\mathbb{R}$. Then their minimal polynomials over $\mathbb{R}$ have degree 2, so there exist irreducible polynomials $g(X) = X^2+aX+b \in \mathbb{R}[X]$ and $h(Y) = Y^2+cY+d \in \mathbb{R}[Y]$ such that $g(x) \in \mathfrak{m}$ and $h(y) \in \mathfrak{m}$. Now since $x^2+y^2=-1$,

$$h(x)+g(y) = ax+cy+(b+d-1) \in \mathfrak{m}\,.$$

If $(a,c) \neq 0$ we can take $f = ax+cy+(b+d-1)$. So assume $(a,c)=(0,0)$. In that case $\mathfrak{m}$ contains the scalar $b+d-1$ so necessarily $b+d=1$. Also $x^2+b, y^2+d \in \mathfrak{m}$. Let $\mathfrak{n}$ be the maximal ideal of $k[X,Y]$ which is the inverse image of $\mathfrak{m}$ under the projection $k[X,Y] \to A$. So we have $X^2+b, Y^2+d \in \mathfrak{n}$. If $b \leq 0$, then $X^2+b$ has a linear factor $X+b'$ which lies in $\mathfrak{n}$ ($\mathfrak{n}$ is prime!), hence $x+b' \in \mathfrak{m}$ and we can take $f = x+b'$. Similarly for $d \leq 0$. So assume both $b$ and $d$ are positive. Then

$$d(X^2+b) - b(Y^2+d) = dX^2 - bY^2 = (\sqrt{d}X - \sqrt{b}Y)(\sqrt{d}X + \sqrt{b}Y)$$

lies in $\mathfrak{n}$, therefore one of its factors, which is of the form $\alpha X + \beta Y$ with $(\alpha,\beta) \neq (0,0)$ lies in $\mathfrak{n}$. So $\alpha x + \beta y \in \mathfrak{m}$.

**4.** Let $A$ be a ring.

(a) Let $\mathfrak{p}$ be a minimal prime ideal of $A$. Show that $\mathfrak{p}A_\mathfrak{p}$ is the nil radical of $\mathfrak{p}A_\mathfrak{p}$. Deduce from this that every element of $\mathfrak{p}$ is a zero divisor in $A$.

We know that in general for a multiplicative subset $S$ of $A$ the prime ideals of $S^{-1}A$ are in one-to-one correspondence with the prime ideals of $A$ that do not intersect $S$. When $S = A - \mathfrak{p}$ this means that the prime ideals of $A_p$ are in one-to-one correspondence with the prime ideals of $A$ contained in $\mathfrak{p}$. Since $\mathfrak{p}$ is minimal, this means that $\mathfrak{p}A_\mathfrak{p}$ is the **only** prime ideal of $A_\mathfrak{p}$, hence equal to the nilradical of $A_\mathfrak{p}$. Nilpotent elements are clearly zero divisors.

(b) Show that if $A$ is reduced, then any zero divisor in $A$ is an element of a minimal prime ideal. Show with an example that this is false if $A$ is not reduced (use Lemma 1.6). See also Corollary 7.1.3(a).

Let $a \in A$ be a zero divisor. So there exists $0 \neq b \in A$ such that $ab = 0$. Since $A$ is reduced, the multiplicative subset $S = \{1, b, b^2, \cdots\}$ does not contain 0 and hence $S^{-1}A$ is nonzero. Note that $a$ maps to 0 under the ring homomorphism $A \to S^{-1}A$ because

$$\frac{b}{1} \cdot \frac{a}{1} = 0$$

and $\frac{b}{1}$ is a unit in $S^{-1}A$. Now let $\mathfrak{q}$ be a minimal prime ideal of $S^{-1}A$ (nonzero rings always have minimal primes by Zorn's lemma). Then $\mathfrak{q} = S^{-1}\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of $A$ which does not intersect $S$ and is minimal among prime ideals with this property. Clearly it follows that $\mathfrak{p}$ is (unconditionally) a minimal prime ideal of $A$. And since $0 \in \mathfrak{q}$, $a$ must lie in $\mathfrak{p}$.

For a counterexample let $k$ be any field and let $A = k[X,Y]/((Y) \cap (X,Y)^2)$. Since $XY \in (Y) \cap (X,Y)^2$ and $X, Y \notin (X,Y)^2$, the element $\overline{X} \in A$ is a zero divisor. We claim that $\overline{X}$ is not in any minimal prime ideal of $A$. For, suppose $\mathfrak{p} \in \operatorname{Spec} A$ contains $\overline{X}$. We can write $\mathfrak{p} = \mathfrak{q} \big/ ((Y) \cap (X,Y)^2)$ for some $\mathfrak{q} \in \operatorname{Spec} k[X,Y]$ such that $(Y) \cap (X,Y)^2 \subseteq \mathfrak{q}$. Note that $Y^2 \in \mathfrak{q}$, so $Y \in \mathfrak{q}$ as $\mathfrak{q}$ is prime. And since $\mathfrak{p}$ contains $\overline{X}$, $\mathfrak{q}$ contains $X$. Thus $\mathfrak{q}$ contains the ideal $(X,Y)$, which is already maximal! So $\mathfrak{q} = (X,Y)$ and $\mathfrak{p} = (\overline{X}, \overline{Y})$. But there is a chain of strict inclusions $(Y) \cap (X,Y)^2 \subsetneq (Y) \subsetneq (X,Y)$. So $(\overline{Y})$ is a prime ideal in $A$ (because $(Y)$ is prime in $k[X,Y]$) which is strictly contained in $\mathfrak{p} = (\overline{X}, \overline{Y})$ thus $\mathfrak{p}$ is not prime.

**5.** Let $k$ be a field. Let $\mathfrak{m}$ be a maximal ideal of $k[T_1, \ldots, T_n]$.

(a) Let $P_1(T_1)$ be a generator of $\mathfrak{m} \cap k[T_1]$ and $k_1 = k[T_1]/(P_1)$. Show that we have an exact sequence

$$0 \to P_1(T_1)k[T_1, \ldots, T_n] \to k[T_1, \ldots, T_n] \to k_1[T_2, \ldots, T_n] \to 0\,.$$

In general if $A$ is a ring and $I$ is an ideal of $A$ and if $X$ is a set of variables, the ring homomorphism between the polynomial rings $A[X] \to (A/I)[X]$ has kernel $I[X] = I \cdot A[X]$. In the question's situation we may take $A = k[T_1]$, $I = (P_1)$ and $X = \{T_2, \ldots, T_n\}$.

(b) Show that there exist $n$ polynomials $P_1(T_1), P_2(T_1, T_2), \ldots, P_n(T_1, \ldots, T_n)$ such that $k[T_1, \ldots T_i] \cap \mathfrak{m} = (P_1, \ldots, P_i)$ for all $i \leq n$. In particular, $\mathfrak{m}$ is generated by $n$ elements.

Let's proceed by induction. The case $n = 1$ follows from the fact that $k[T_1]$ is a PID. Now assume the claim holds for $n - 1$. By (a), there exists a polynomial $P_1(T_1) \in k[T_1]$ such that $k[T_1] \cap \mathfrak{m} = (P_1)$ and writing $k_1 = k[T_1]/(P_1)$, we have a surjective ring homomorphism

$$\varphi : k[T_1, \ldots, T_n] \to k_1[T_2, \ldots, T_n]$$

with kernel $P_1(T_1)k[T_1, \ldots, T_n]$. First, note that the ideal $(P_1)$ of $k[T_1]$ is the inverse image of $\mathfrak{m}$ under the inclusion $k[T_1] \hookrightarrow k[T_1, \ldots, T_n]$ so is a prime ideal. But $k[T_1]$ is a PID, so $(P_1)$ is maximal. Thus $k_1$ is a field.

Second, since $P_1 \in \mathfrak{m}$, the ideal $P_1(T_1)k[T_1, \ldots, T_n]$ is contained in $\mathfrak{m}$. So $\mathfrak{m}_1 = \varphi(\mathfrak{m})$ is a maximal ideal of $k_1[T_2, \ldots, T_n]$.

Now by the induction hypothesis there exist $n-1$ polynomials $Q_2(T_2)$, $Q_3(T_2, T_3)$, ..., $Q_n(T_2, \dots T_n)$ such that $k_1[T_2, \dots, T_i] \cap \mathfrak{m}_1 = (Q_2, \dots, Q_i)$ for all $2 \leq i \leq n$.

By definition, for every $2 \leq i \leq n$, the ring homomorphism $\varphi$ maps $k[T_1, \dots, T_i]$ onto $k_1[T_2, \dots, T_i]$ with kernel $P_1(T_1)k[T_1, \dots, T_i]$. Therefore there exists $P_i(T_1, \dots, T_i)$ such that $\varphi(P_i) = Q_i$. Consider the restricted ring homomorphism

$$\varphi_i : k[T_1, \dots, T_i] \to k_1[T_2, \dots, T_i] .$$

Chasing the commutative square formed by restricting $\varphi$ to $\varphi_i$, we see that

$$\begin{aligned} k[T_1, \dots, T_i] \cap \mathfrak{m} &= {\varphi_i}^{-1}(k_1[T_2, \dots, T_i] \cap \mathfrak{m}_1) \\ &= {\varphi_i}^{-1}(Q_2, \dots, Q_i) \\ &= (P_1, P_2, \dots, P_i) . \end{aligned}$$

**8.** Let $\varphi : A \to B$ be an integral ring homomorphism.

(a) Show that $\operatorname{Spec} \varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ maps a closed point to a closed point, and that any preimage of a closed point is a closed point.

Let $\mathfrak{n}$ be a maximal ideal of $B$. Then $\varphi$ induces an injective ring homomorphism

$$\psi : A/\varphi^{-1}(\mathfrak{n}) \hookrightarrow B/\mathfrak{n} .$$

Since $\varphi$ is integral, so is $\psi$. But $B/\mathfrak{n}$ is a field and by the following lemma $A/\varphi^{-1}(\mathfrak{n})$ is a field. Hence $\varphi^{-1}(\mathfrak{n})$ is maximal in $A$.

**Lemma 1.** *If $B$ is a integral domain with a subring $A$ such that the inclusion $A \subseteq B$ is integral, then $A$ is a field if and only if $B$ is a field.*

*Proof.* Suppose $B$ is a field and let $a \in A \setminus \{0\}$. Since $a^{-1} \in B$ is integral over $A$, there exists $b_0, \dots, b_{n-1} \in A$ such that

$$\begin{aligned} 0 &= b_0 + b_1 a^{-1} + \cdots b_{n-1}(a^{-1})^{n-1} + (a^{-1})^n \\ &= b_0 + b_1 a^{-1} + \cdots b_{n-1} a^{-n+1} + a^{-n} \\ a^{-n} &= -(b_0 + b_1 a^{-1} + \cdots + b_{n-1} a^{-n+1}) \end{aligned}$$

so multiplying both sides by $a^{n-1}$ yields

$$a^{-1} = -(b_0 a^{n-1} + b_1 a^{n-2} + \cdots + b_{n-1}) \in A .$$

Suppose $A$ is a field and let $b \in B \setminus \{0\}$. Let $A[b]$ be the subring of $B$ generated by $A$ and $b$. Then there is a surjection $\epsilon : A[X] \to A[b]$ that sends $X$ to $b$. Since $b$ is integral over $A$, $\ker \epsilon \neq 0$. And since $A[b]$ is a domain $\ker \epsilon$ is a prime ideal in $A[X]$. But $A[X]$ is a PID, so $0 \neq \ker \epsilon$ must be a maximal ideal. Thus

$$A[b] \cong A[X]/\ker \epsilon$$

is a field. Thus $b$ has an inverse $b^{-1}$ in $A[b]$ and hence in $B$. □

The second part requires showing that given a prime ideal $\mathfrak{q}$ in B such that $\varphi^{-1}(\mathfrak{q})$ is maximal in $A$, then $\mathfrak{q}$ is maximal. Again using the injection $A/\varphi^{-1}(\mathfrak{q}) \hookrightarrow B/\mathfrak{q}$ and the lemma (in the other direction this time) we get that $B/\mathfrak{q}$ is a field hence $\mathfrak{q}$ is maximal.

(b) Let $\mathfrak{p} \in \operatorname{Spec} A$. Show that the canonical homomorphism $A_\mathfrak{p} \to B \otimes_A A_\mathfrak{p}$ is integral.

We will show that in general for a multiplicative set $S$, the ring homomorphism $S^{-1}A \to S^{-1}B$ is integral, which is enough since $B \otimes_A S^{-1}A \cong S^{-1}B$ in a natural way. And indeed given $b \in B$ and $s \in s$, since $b$ is integral over $A$ we have

$$a_0 + a_1 \cdot b + a_2 \cdot b^2 + \cdots a_{n-1} b^{n-1} + b^n = 0$$

for some $a_0, \ldots, a_{n-1} \in A$. Therefore in $S^{-1}B$, we have

$$\begin{aligned} 0 &= \frac{a_0 + a_1 \cdot b + a_2 \cdot b^2 + \cdots a_{n-1} b^{n-1} + b^n}{s^n} \\ &= \frac{a_0}{s^n} + \frac{a_1}{s^{n-1}} \cdot \frac{b}{s} + \frac{a_2}{s^{n-2}} \cdot \left(\frac{b}{s}\right)^2 + \cdots + \frac{a_{n-1}}{s} \cdot \left(\frac{b}{s}\right)^{n-1} + \left(\frac{b}{s}\right)^n \end{aligned}$$

which shows that $b/s$ is integral over $S^{-1}A$.

(c) Let $T = \varphi(A \setminus \mathfrak{p})$. Let us suppose that $\varphi$ is injective. Show that $T$ is a multiplicative subset of $B$, and that $B \otimes_A A_\mathfrak{p} = T^{-1}B \neq 0$. Deduce from this that $\operatorname{Spec} \varphi$ is surjective if $\varphi$ is integral and injective.

Being a ring homomorphism, $\varphi$ maps the multiplicative subset $A \setminus \mathfrak{p}$ in $A$ to a multiplicative subset of $B$, which is called $T$ here. Since the functor $- \otimes_A A_\mathfrak{p}$ is exact, $A \otimes_A A_\mathfrak{p} = A_\mathfrak{p}$ injects in $B \otimes_A A_\mathfrak{p}$ because $A$ injects in $B$. As $A_\mathfrak{p} \neq 0$, we deduce that $B \otimes_A A_\mathfrak{p} \neq 0$.

We want to show $\operatorname{Spec} \varphi$ is surjective. So let $\mathfrak{p}$ be a prime ideal in $A$ and write $T = \varphi(A \setminus \mathfrak{p})$ as above. Since $T^{-1}B = B \otimes_A A_\mathfrak{p} \neq 0$, $T^{-1}B$ has a maximal ideal which is necessarily of the form $T^{-1}\mathfrak{q}$ for a prime ideal $\mathfrak{q}$ of $B$ such that $\mathfrak{q} \cap T = \emptyset$. Now $\varphi_\mathfrak{p} : A_\mathfrak{p} \to B \otimes_A A_\mathfrak{p}$ is integral by (b), so by (a) $f^{-1}(T^{-1}\mathfrak{q})$ is a maximal ideal in $A_\mathfrak{p}$. But there is only one maximal ideal in $A_\mathfrak{p}$, so ${\varphi_\mathfrak{p}}^{-1}(T^{-1}\mathfrak{q}) = \mathfrak{p}A_\mathfrak{p}$. Taking the inverse image of $T^{-1}\mathfrak{q}$ in two different ways given by the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ A_\mathfrak{p} & \xrightarrow{\varphi_\mathfrak{p}} & B \otimes_A A_p \end{array}$$

yields $\operatorname{Spec}(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$.

**9.** Let $A$ be a finitely generated algebra over a field $k$.

(a) Let us suppose that $A$ is finite over $k$. Show that $\operatorname{Spec} A$ is a finite set, of cardinality bounded from above by the dimension $\dim_k A$ of $A$ as a vector space (Construct a strictly descending chain of ideals by taking intersections of maximal ideals). Show that every prime ideal of $A$ is maximal.

For a given $A$-module $M$, denote its isomorphism class by $[M]$. There is a bijection

$$\begin{aligned} \{\text{maximal ideals of } A\} &\leftrightarrow \{\text{isomorphism classes of simple } A\text{-modules}\} \\ \mathfrak{m} &\mapsto [A/\mathfrak{m}] \\ \operatorname{ann}_A(S) &\leftarrow\!\shortmid [S] \end{aligned}$$

(commutativity of $A$ is crucial here). The assumption on $A$ says that $A$ is a finite-dimensional $k$-algebra hence an artinian ring. Therefore $A$ has finitely many isomorphism classes of simple modules, hence finitely many maximal ideals. Let $S_1, \dots, S_r$ be a set of representatives for simple $A$-modules. Since $A/\operatorname{Rad} A$ is semisimple,

$$A/\operatorname{Rad} A \cong S_1^{n_1} \oplus \cdots S_r^{n_r}$$

for some $n_1, \dots n_r$. In particular

$$\dim_k A = \dim_k \operatorname{Rad} A + \sum_{j=1}^{r} n_j \cdot \dim_k S_j \geq r \, .$$

Thus we showed that the number of maximal ideals in $A$ is less than or equal to $\dim_k A$. To finish, we show that every prime is maximal. Let $\mathfrak{p} \in \operatorname{Spec} A$. Then $A/\mathfrak{p}$ is an artinian integral domain. But artinian domains are fields (consider the descending chain $(a) \supseteq (a^2) \cdots$ for any nonzero $a$), hence $\mathfrak{p}$ is maximal.

(b) Show that $\operatorname{Spec} k[T_1, \dots T_d]$ is infinite if $d \geq 1$.

Note that $B := k[T_1, \dots, T_d]$ is a UFD. So irreducible elements are prime, that is, they generate prime ideals. So it is enough to show that there are finitely many irreducible elements. The existence of such elements is guaranteed by the condition $d \geq 1$, as it tells us that $B$ is a noetherian ring which is not a field and hence the collection of proper principal ideals has a maximal element which is not zero (recall that $p \in A$ irreducible $\Leftrightarrow$ $(p)$ is maximal among proper principal ideals). The old and beautiful Euclidean proof for the infinitude of the primes does the rest.

(c) Show that $\operatorname{Spec} A$ is finite if and only if $A$ is finite over $k$.

One direction was established in (a). Suppose $\operatorname{Spec} A$ is finite. By the Noether normalization lemma, there exists an integer $d \geq 0$ such that there is an injective finite $k$-algebra homomorphism $\varphi : k[T_1, \dots, T_d] \hookrightarrow A$. Being finite, $\varphi$ is integral. Therefore by part (c) of the previous question,

$$\operatorname{Spec}(\varphi) : \operatorname{Spec} A \to \operatorname{Spec} k[T_1, \dots, T_d]$$

is surjective. Thus $\operatorname{Spec} k[T_1, \dots, T_d]$ is finite. But then part (b) forces $d = 0$. Hence A is finite over $k$ via $\varphi$.