MATH 4567, FALL 2023
HOMEWORK PROBLEMS No.2
Due on October 25

Problem 1. Let $S_n(x)$ denote the n-th partial sum of the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

in the interval $-\pi \le x \le \pi$ for the function

$$f(x) = \begin{cases} 2x - 3, & \text{for} \quad x < 0, \\ x + 1, & \text{for} \quad x > 0. \end{cases}$$

a) Evaluate for each $x \in [-\pi, \pi]$ the limit $S(x) = \lim_{n\to\infty} S_n(x)$.
b) Sketch the graph of S on the whole real line.
c) Find the values $S(10)$, $S(20)$.

Hint: The function S is 2π-periodic, so it is enough to know its values on $[-\pi, \pi)$.

Problem 2. *a*) In the interval $-\pi \le x \le \pi$ find the Fourier series for the function

$$f(x) = |\sin(x)|.$$

b) Is it true that the n-th partial sums $S_n(x)$ of that Fourier series are convergent to $f(x)$ at all points $x \in \mathbf{R}$? Explain.
c) If so, is it true that the convergence is uniform, that is,

$$\sup_{x\in\mathbf{R}} |S_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty\,?$$

Hint: Check whether or not the function f is (2π)-periodic and has a piecewise continuous derivative in $[-\pi, \pi]$.

Problem 3. Check that the function $f(x) = -\log x$ belongs to $L^2[0,1]$ and find its L^2-norm. Then consider its cosine Fourier series in the interval $0 \le x \le 1$, that is,

$$-\log x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\pi n x).$$

Evaluate the sum $\sum_{n=1}^{\infty} a_n^2$ without computing the coefficients a_n.

Hint: Apply Parseval's equality for the cosine Fourier series in $[0, c]$. You will also need to compute a_0.

Problem 4. Using elementary arguments, solve the boundary value problem

$$u_{xx} = -y^2 \cos x, \qquad u = u(x,y), \ \ 0 \le x \le \pi,$$
$$u(0,y) = y^2,$$
$$u(\pi, y) = \pi \sin y - y^2.$$