MATH 4567, FALL 2023
HOMEWORK ASSIGNMENT No.4
Due on December 6 (Wednesday)

Problem 1. Given a continuous function $f(x)$ on $[0,\pi]$, solve the temperature problem:

$$\begin{aligned} &u_t = ku_{xx}, \qquad u = u(x,t),\ \ 0 \le x \le \pi,\ \ t > 0\ \ (k > 0)\\ &u_x(0,t) = u(0,t),\\ &u_x(\pi,t) = u(\pi,t),\\ &u(x,0) = f(x). \end{aligned}$$

Write your answer in the form of an infinite functional series

$$u(x,t) = \sum_{n=0}^{\infty} c_n y_n(x) T_n(t).$$

Describe the functions $y_n(x)$ and $T_n(t)$ that are involved and indicate how to compute the coefficients c_n in terms of f.

Note. You will need to consider an associated Sturm-Liouville problem which is exactly the same as the one in Problem 4 from HW3 with parameters $\beta = 1$ and $c = \pi$.

Problem 2. Given parameters A, B, C (real), consider the temperature problem with non-homogeneous boundary conditions:

$$\begin{aligned} &u_t = ku_{xx}, \qquad u = u(x,t),\ \ 0 \le x \le \pi,\ \ t > 0\ \ (k > 0)\\ &u_x(0,t) = u(0,t) + A,\\ &u_x(\pi,t) = u(\pi,t) + B,\\ &u(x,0) = Cx. \end{aligned}$$

Reduce it to Problem 1 by virtue of a suitable substitution $u(x,t) = U(x,t) + \Phi(x)$. Indicate new initial temperatures $F(x)$ in the homogeneous problem about $U(x,t)$.

Problem 3. Use the Fourier transform to solve the temperature problem in the upper half-plane

$$\begin{aligned} &u_t = ku_{xx}, \qquad u = u(x,t),\ \ -\infty < x < \infty,\ \ t \ge 0,\quad u \text{ is bounded},\\ &u(x,0) = e^{-2x^2}. \end{aligned}$$

Hint. First formulate a general theorem about the boundary value problems

$$\begin{aligned} &u_t = k\,u_{xx}, \qquad u = u(x,t),\ \ -\infty < x < \infty,\ \ t \ge 0,\quad u \text{ is bounded},\\ &u(x,0) = f(x). \end{aligned}$$

Problem 4. Use the Fourier transform to solve the boundary value problem, involving the Laplace equation:

$$\begin{aligned} &\Delta u = 0, \qquad u = u(x,y),\ \ x, y \ge 0,\quad u \text{ is bounded},\\ &u_x(0,y) = 0,\\ &u(x,0) = \tfrac{1}{1+x^2}. \end{aligned}$$

Hint. First formulate a general theorem about boundary value problems of the form

$$
\begin{aligned}
&\Delta u = 0, \qquad\qquad && u = u(x, y), \;\; x, y \geq 0, \quad u \text{ is bounded,} \\
&u_x(0, y) = 0, && \\
&u(x, 0) = f(x). &&
\end{aligned}
$$

Note that the Fourier transforms for the functions

$$f(x) = e^{-x^2/(2\sigma^2)} \quad \text{and} \quad f(x) = \frac{1}{1 + x^2}$$

(needed in Problems 3 and 4) are known and have been evaluated in class.