Changes of variables: First linear changes.

Given $T: \mathbb{R}^n \to \mathbb{R}^n$ invertible linear map, if $f: \mathbb{R}^n \to \mathbb{R}$ integrable, then $f \circ T$ is integrable and

$$\int_{\mathbb{R}^n} f(\underline{y})\, |d^n \underline{y}| = |\det T| \cdot \int_{\mathbb{R}^n} f(T(\underline{x}))\, |d^n \underline{x}|$$

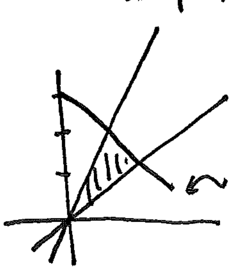
(where $T: \underline{x} \mapsto \underline{y}$)

pf: RHS is limit of Riemann sums:

$$\lim_{N\to\infty} \sum_{C \in D_N(\mathbb{R}^n)} \sup_C (f\circ T) \cdot \underbrace{\underset{\substack{\text{constant}\\ \text{so we removed}\\ \text{it from sup.}}}{|\det T|}\, \overset{\mathrm{vol}_n(C)}{|\cancel{d^n x}|}}_{\mathrm{vol}_n(T(C))}$$

$$= \lim_{N\to\infty} \sum_{P \in \underset{\text{new paving}}{T(D_N)}} \sup_P (f)\ \mathrm{vol}_n(P) = \text{LHS}. \qquad //$$

Example: Compute $\int_{\mathbb{R}^2} (x_1 + x_2)\, |d^2(x_1, x_2)|$ over triangle:



Idea: Map line $x_1 = x_2$ to new coords (u_1, u_2) with $u_2 = 0$.

$u_2 = x_1 - x_2$

$u_1 = x_1 + x_2$.

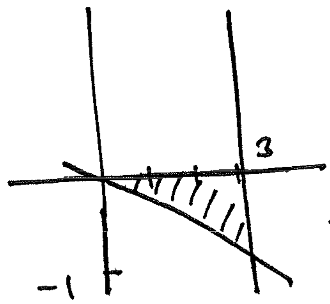
$$S: \begin{pmatrix} 1 & +1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} +1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

then linear change of vars :

$$\int_{\text{triangle}} (x_1 + x_2) \, |d^2 \underline{x}| = \frac{1}{2} \int_{\substack{\text{new} \\ \text{triangle}}} u_1 \, d(u_1, u_2)$$

where new triangle :



$S(x_1 = x_2)$

$= \text{line}(u_2 = 0)$

$S(x_1 + x_2 = 3) = \text{line}(u_1 = 3)$

$S(2x_1 = x_2) = \text{line}\left(2 \cdot \underbrace{\left(\frac{u_1 + u_2}{2}\right)} = \frac{u_1 - u_2}{2}\right)$

$u_1 + u_2 = \frac{u_1}{2} - \frac{u_2}{2}$

$2(u_1 + u_2) = u_1 - u_2$

$u_2 = -\frac{1}{3} u_1$

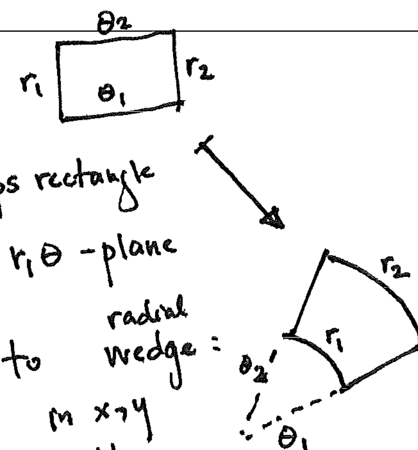
Polar coordinates : (change in $\mathbb{R}^2$)

(not a linear change)

$$p : \begin{pmatrix} r \\ \theta \end{pmatrix} \longmapsto \begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix}$$

$$\begin{pmatrix} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{pmatrix} \longleftarrow \begin{pmatrix} x \\ y \end{pmatrix} : p^{-1}$$

p : maps rectangle in r, θ -plane to radial wedge in x, y plane :



Proposition :

$$\int_{A \subseteq \mathbb{R}^2} f(x, y) \, |d(x, y)| = \int_{p^{-1}(A)} f(r\cos\theta, r\sin\theta) \, r \, |d(r, \theta)|$$

Intuition : $r \, |d(r, \theta)|$ replacing $\det(p) \, |dr, \theta|$ in linear case.

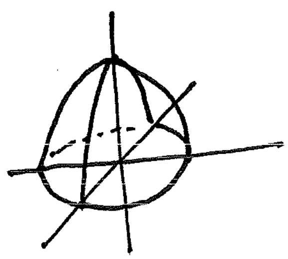
For non-linear maps, replace det(P) by the Jacobian (which may vary depending on r, θ)

$$p : \begin{pmatrix} r \\ \theta \end{pmatrix} \longmapsto \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

with Jacobian $\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$ with det $= r$.

prove a general formula on Friday. For today, do example.

<u>Example</u>: volume in dome $z = \begin{cases} 4 - x^2 - y^2 & \text{if } |(x,y)| \le 2 \\ 0 & \text{else.} \end{cases}$



Common theme with polar coords:
if domain or integrand involve trig. quantities e.g. x^2+y^2, or parametrized curves best given in polar coords. like lemniscate example in book

try changing to polar.

$$\int_{\substack{\text{circle} \\ \text{of radius} \\ 2}} (4 - x^2 - y^2)\, d(x,y) = \int_{\substack{\text{rectangle} \\ [0,2]\times[0,2\pi]}} (4 - r^2)\, r\, dr\, d\theta$$

$$\overset{\text{Fubini}}{=} \int_0^2 \int_0^{2\pi} (4r - r^3)\, dr\, d\theta$$

$$= 2\pi \int_0^2 (4r - r^3)\, dr = 2\pi \left[2r^2 - \frac{1}{4}r^4\right]\Big|_0^2$$

$$= 2\pi\,[4 - 0]$$

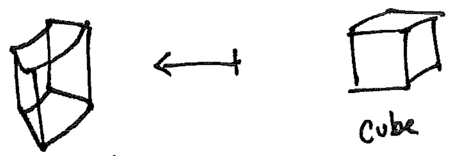
$$= \boxed{8\pi}$$

Two ways of generalizing polar coords to $\mathbb{R}^3$ – cylindrical, spherical coordinates.

Never guess what they are best at measuring.

Cylinder: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}$ $\Big\}$ polar in plane

with Jacobian $\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\text{polar Jac.}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$



cylindrical wedge $\qquad$ cube

$\begin{vmatrix} x \\ y \\ z \end{vmatrix} \longleftarrow (r, \theta, z)$ $\qquad = r\,|d(r,\theta,z)|.$

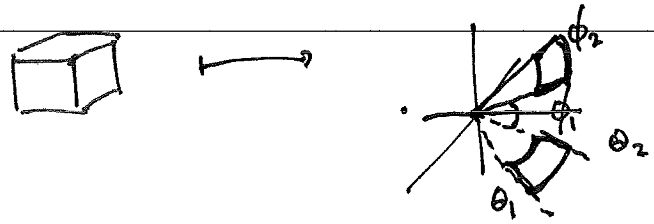
Spherical:

$$\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} \longmapsto \begin{pmatrix} x = r\cos\theta\cos\phi \\ y = r\sin\theta\cos\phi \\ z = r\sin\phi \end{pmatrix}$$

in our book ϕ is angle with line in x-y plane, not from north pole

their Jacobian:

$$r^2\cos\phi\;|d(r,\theta,\phi)|$$



$\phi \in [-\pi/2, \pi/2]$

$\theta \in [0, 2\pi]$

$r \in [0, \infty)$