On Wednesday, defined integration using Riemann sums on dyadic pavings: If g is integrable then $g: \mathbb{R}^n \to \mathbb{R}$

$$\int_{\mathbb{R}^n} g \, |d^n \underline{x}| = \lim_{N\to\infty} \sum_{\substack{\text{n-cubes in} \\ \text{dyadic paving of} \\ \text{level } N}} g(x_i^*) \, \underbrace{\mathrm{vol}}(i^{th} \text{ cube})$$

(sides: $1/2^N$)

same constant for all cubes : $(1/2^N)^n$

Recall these are half-closed cubes so disjoint. (draw n=2 again)

$$= \lim_{N\to\infty} \left(\frac{1}{2^N}\right)^n \cdot \sum_{\substack{\text{cubes intersecting} \\ \text{support of } g}} g(x_i^*)$$

x_i^* : min of g on ith cube (Lower sum) — an increasing function of N

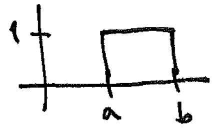
max of g on ith cube (Upper sum) — decreasing fun. of N.

or any other point in cube since, for any x_i^*

$$g(x_i^{min}) \le g(x_i^*) \le g(x_i^{max})$$

Easiest functions to integrate : constants.

$$\int_a^b 1 \, dx \underset{FTC}{=} x \Big|_a^b = b-a$$

$\underset{\text{area}}{=}$  $= b-a$

roughly: $(b-a)2^N - 1$ cubes. True if $b-a$ is integer, for example.

$$\underset{\text{Riemann sum}}{=} \lim_{N\to\infty} \frac{1}{2^N} \sum_{i=0} \underbrace{g(x_i^*)}_{\substack{\text{"} \\ 1 \text{ no matter the } x_i^*}} = b-a.$$

Now write it as : $\displaystyle\int_{\mathbb{R}} 1_{[a,b]} \, |dx|$ where $1_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a,b] \\ 0 & \text{else.} \end{cases}$

"characteristic function of the set $[a,b]$"

("indicator function" in book)

In higher dimensions, given a set A,

$\displaystyle\int_{\mathbb{R}^n} 1_A(\underline{x}) \, |d^n \underline{x}|$ is very interesting!

already much cooler for $n=2$.

In fact, some sets A have 1_A not integrable, even in $\mathbb{R}^1$.

e.g. $A =$ irrationals in $[0,1]$.

Book says a set A is "pavable" if 1_A is integrable.

Basic facts about $\text{volume}(A) := \int_{\mathbb{R}^n} 1_A(\underline{x}) \, |d^n \underline{x}|$:

Theorem: If A, B disjoint, pavable, then $A \cup B$ is pavable

and $\text{vol}(A \cup B) = \text{vol}(A) + \text{vol}(B)$.

pf: First write out definitions! $\text{vol}(A \cup B) = \int_{\mathbb{R}^n} 1_{A \cup B} \, |d^n \underline{x}|$

$\text{vol}(A) = \int_{\mathbb{R}^n} 1_A \, |d^n \underline{x}|$; $\text{vol}(B) = \int_{\mathbb{R}^n} 1_B \, |d^n x|$

In section, you proved with Theo that

$$\int_{\mathbb{R}^n} f + g = \int_{\mathbb{R}^n} f + \int_{\mathbb{R}^n} g$$ (squeeze theorem with upper/lower Riemann sums.)

so result follows upon noting that $1_{A \cup B} = 1_A + 1_B$ if A, B disjoint.

Application: Volume is translation invariant. Given set A (pavable), vector $\underline{v}$

then $\text{vol}(A + \underline{v}) = \text{vol}(A)$. ← in particular $A + \underline{v}$ pavable.

proof: To show $A + \underline{v}$ integrable, need to compute upper, lower Riemann sums.

What is $L(1_{A+\underline{v}})$? $\lim_{N \to \infty} \left(\frac{1}{2^N}\right)^n \cdot \sum_{\substack{\text{cubes} \\ \text{in} \\ A+\underline{v}}} 1$

Since L_N is increasing in $N \to \infty$ we have

we mean: cubes entirely contained in $A + \underline{v}$

$$L(1_{A+\underline{v}}) \geq L\Big(1_{\substack{\text{Union of}\\ \text{level } N \text{ cubes}\\ \text{in } A+\underline{v}}}\Big) = L\Big(\sum_{\substack{C:\text{ cube}\\ \text{entirely}\\ \text{in } A\\ \text{of level } N}} 1_{C+\underline{v}}\Big)$$

$C+\underline{v}$ is again a dyadic cube. This is integrable with volume $= \mathrm{vol}(C) = \left(\frac{1}{2^N}\right)^n$

So $$L\Big(\sum 1_{C+\underline{v}}\Big) = \int \sum 1_{C+v} = \sum \int 1_{C+v}$$

$$= \sum \mathrm{vol}_n(C)$$

(sum over cubes in A at each stage)

So $$L(1_{A+\underline{v}}) \geq \sum_{\substack{\text{level } N\\ \text{cubes}\\ C}} \mathrm{vol}_n(C) \quad \text{for each } N$$

$$\Rightarrow \quad L(1_{A+\underline{v}}) \geq \lim_{N\to\infty} \sum_{\substack{\text{level } N\\ \text{cubes } C\\ \text{in } A}} \mathrm{vol}_n(C) = L(1_A).$$

Similarly show upper bound for $\mathcal{U}(1_{A+v})$ using cubes with $\neq \phi$ intersection with A.

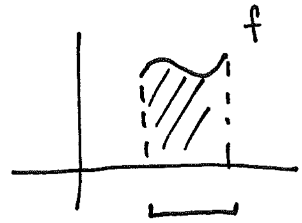
then get $$L(1_A) \leq L(1_{A+\underline{v}}) \leq \mathcal{U}(1_{A+\underline{v}}) \leq \mathcal{U}(1_A).$$

But outer two quantities are equal since A assumed pavable.

volumes are important application but we can, of course, consider more interesting integrable functions other than characteristic functions.

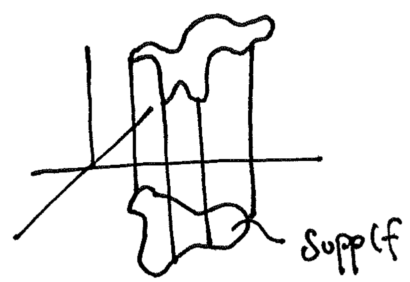
Two viewpoints: Given f,

① $\displaystyle\int_{\mathbb{R}} f \, |d^1 x|$ =



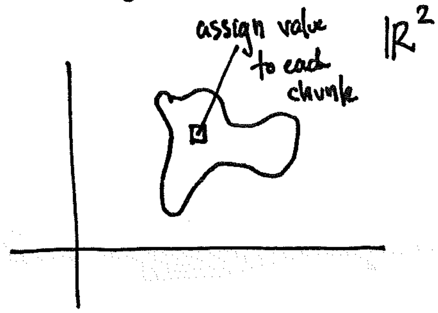
"area below f"

or $\displaystyle\int_{\mathbb{R}^2} f \, |d^2 \underline{x}|$ =



"volume beneath surface f"

② As weighting factor



"density function" where Supp(f) is our material and f is density at any point.

Physical applications: $A \subset \mathbb{R}^n$.

μ : density function

$$\text{Mass}(A) = \int_A \mu(\underline{x}) \, |d^n \underline{x}| \overset{\text{OR}}{=} \int_{\mathbb{R}^n} \mu(\underline{x}) \cdot \mathbf{1}_A(\underline{x}) \, |d^n \underline{x}|$$

Center of gravity of A (or center of mass) in direction x_i in $\mathbb{R}^n$ $= \displaystyle\int_A x_i \, \mu(\underline{x}) \, |d^n \underline{x}|$.