On Friday, reminded you about two ways of defining smooth k-manifold.

1. locally as zero locus of $F : \mathbb{R}^n \to \mathbb{R}^{n-k}$

2. globally as parametrization $\gamma : \mathcal{U} \subset \mathbb{R}^k \to M \subset \mathbb{R}^n$.

example: helix $\gamma : t \in \mathbb{R} \mapsto \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix} \in \mathbb{R}^3$

(parametric curve)

defines smooth 1-manifold.

(Jacobian matrix not $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\forall\, t$.)

So $D\gamma$ is one-one, for all t.

<u>Plan</u>: Use parametrizations to compute volumes of manifolds (in our example, 1-dim'l volume should be arc length of (part of) helix inside $\mathbb{R}^3$.)

define
$$\mathrm{vol}_k(M) = \int_{\gamma(\mathcal{U})} |d^k x|$$

$$= \int_{\mathcal{U}} \mathrm{vol}_k\big(P_{\gamma(\underline{u})}(D_1\gamma(\underline{u}), \ldots, D_k\gamma(\underline{u}))\big)\, |d^k \underline{u}|$$

Better: Riemann sums like

$$\lim_{N \to \infty} \sum_{C \in \mathcal{D}_N(\mathbb{R}^k)} \mathrm{vol}_k(\gamma(C \cap \mathcal{U})) \approx \lim_{N \to \infty} \sum_{C \in \mathcal{D}_N(\mathbb{R}^k)} \mathrm{vol}_k(\mathcal{U}) \cdot \mathrm{vol}_k(P_{\gamma(\underline{u})}(\ldots))$$

Issue #1: What is k-volume of k-parallelogram spanned by k vectors $v_1, \dots, v_k$ in $\mathbb{R}^n$?

Not $\left| \det(v_1, \dots v_k) \right|$, since doesn't make sense.

Trickier: $\sqrt{\det(\underbrace{T^T T})}$ where $T = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{pmatrix}$

matrix is square.

T is $n \times k$.

T^T is $k \times n$

so $T^T T$ is $k \times k$.

3×2 example: $T = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \\ v_{3,1} & v_{3,2} \end{pmatrix}$ (two vectors in $\mathbb{R}^3$ spanning plane)

then $\det(T^T T) = \det \begin{pmatrix} - v_1 - \\ - v_2 - \end{pmatrix} \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \det \begin{pmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 \\ v_2 \cdot v_1 & v_2 \cdot v_2 \end{pmatrix}$

all dot products: $v_1 \cdot v_1 = |v_1|^2$

$v_i \cdot v_j = |v_i||v_j| \cos\theta$ $\quad \theta$: angle between v_i, v_j in plane spanned by them.

independent of "anchor" coordinates

- location of k-parallelogram in space.

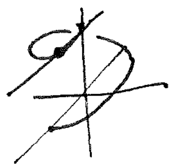
also independent of embedding of $\mathbb{R}^k$ in $\mathbb{R}^n$.

Back to our example of helix: $\gamma : t \longmapsto \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}$ then $D\gamma = \begin{bmatrix} \frac{d}{dt}(\cos t) \\ \frac{d}{dt}(\sin t) \\ \frac{d}{dt}(t) \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$

$D\gamma(t)^T D\gamma(t) = \gamma'(t) \cdot \gamma'(t)$

$= \sin^2 t + \cos^2 t + 1 = 2.$

so arc length of one rotation of helix : $\displaystyle\int_0^{2\pi} \sqrt{2}\, dt = 2\pi \cdot \sqrt{2}.$



(or up to any height $T = 2\pi T$.)

Issue # 2 : Very hard to find perfect parametrization ($\mathcal{U}$ open, with $\gamma : \mathcal{U} \to M$ one-one, onto, ...)

So we relax definition as in p. 3 of Friday's notes

(0) $\partial \mathcal{U}$ has k-dim'l volume 0

(1) $\gamma(\mathcal{U}) \supset M$

(2) $\exists\ X \subset \mathcal{U}$ with k-dim'l vol. 0 s.t. $\gamma(\mathcal{U} - X) \subset M$.

(3) $\gamma : \mathcal{U} - X \to M$ is one-one, C^1 function with locally Lipschitz derivative.

(4) $D\gamma(\underline{u})$ is one-one $\forall\ \underline{u} \in \mathcal{U} - X$.

(5) $\gamma(X) \cap C$ has k-volume 0 $\forall$ compact $C \cap M$.

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basic idea : pushed all "trouble spots" in failing parametrizations into a set of vol. 0, X, In examples, X usually includes $\partial \mathcal{U}$. (check that circle can be parametrized using relaxed parametrization)

Example: parametrizing a cone. Cone in $\mathbb{R}^3$ given by $x^2+y^2-z^2=0$.

(technically not a manifold. Not graph at origin)

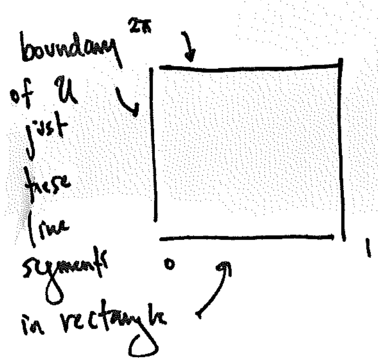
associate manifold $M = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x^2+y^2-z^2=0, \; z \in (0,1) \right\}$

How to parametrize it?

$$\gamma : \begin{pmatrix} r \\ \theta \end{pmatrix} \longmapsto \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ r \end{pmatrix}$$

$$\underbrace{[0,1]\times[0,2\pi]}_{U} \longrightarrow \mathbb{R}^3$$

Set $X = \partial U$ so that $U - X = (0,1)\times(0,2\pi)$



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claim: their volume is 0.

(~~Monday~~ Wednesday, make definition of volume 0 which ensures that if

$0 \le m < k \le n$ then

k-volume of m-manifold is 0.)

If $n \times 2$ example: $T = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$ with $\underline{v_1} = (v_{1,1}, v_{2,1}, \dots, v_{n,1})$

$\underline{v_2} = (v_{1,2}, v_{2,2}, \dots, v_{n,2})$

then regardless of n, $\det(T^T T) = \det \begin{pmatrix} v_1\cdot v_1 & v_1 \cdot v_2 \\ v_2 \cdot v_1 & v_2 \cdot v_2 \end{pmatrix}$

$$= |v_1|^2 |v_2|^2 - (\underbrace{v_1\cdot v_2}_{|v_1|^2|v_2|^2\cos^2\theta})^2 = |v_1|^2|v_2|^2 (\underbrace{1-\cos^2\theta}_{\sin^2\theta})$$

Square root gives familiar form of $\parallel$-gram vol.