New chapter - integration to produce signed volume where sign $\pm$ depends on orientation. On curve, two clear choices of orientation: e.g. circle has clockwise / counterclockwise. What about surface? Less clear. Roughly - linearize to consider tangent hyperplanes at each point, and attach orientation to tangent hyperplanes (i.e. k-vector spaces in $\mathbb{R}^n$) according to choice of basis. Subtle.

Also need new objects to integrate, that are responsive to this orientation. With absolute volume $|d^k \underline{x}|$ or $|\det T| \cdot |d^k \underline{x}|$ and $|\det(T)|$ not sensitive to swapping of rows.

Some fixes: ① $\det(T)$ is sensitive to swapping of rows. changes by -1 as we'd like.

② if manifold is parametrized: $\gamma: U \to M$ then γ indicates an orientation. Think again about unit circle: $\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$: counterclockwise

$\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$: clockwise. Turns out same is true in higher dimensions.

Expanding on ①, what is it about det that makes it respect orientation is antisymmetric. (also want it to be linear in each component.)

Definition: A k-form on $\mathbb{R}^n$ is a function

$$\underbrace{\phi : (\mathbb{R}^n)^k}_{k \text{ vectors in } \mathbb{R}^n} \longrightarrow \mathbb{R} \quad \text{with}$$

- linear in each factor
- antisymmetric.

if $k = n$, then these are precisely our earlier defining properties of determinant.

if $k > n$, then any k vectors are linearly dependent.

write, say $\vec{v}_k = \sum_{i=1}^{k-1} c_i \vec{v}_i$. Then k-form takes

$$\phi(v_1, \dots, v_k) = \phi(v_1, \dots, v_{k-1}, \sum c_i v_i) \underset{\text{linearity}}{\overset{\uparrow}{=}} \sum_i c_i \underbrace{\phi(v_1, \dots, v_{k-1}, v_i)}_{=\,0}$$

0 since anti-~~sym~~ symmetry says switching ith position and kth position produces neg. sign. On other hand, both have v_i's as component.

So there are no nonzero k forms if $k > n$.

if $k < n$, might try

$$\sqrt{\det (v_1, \dots, v_k)^T (v_1 \cdots v_k)}.$$

Doomed to fail since always positive so can't by antisymmetric.

(also fails to be linear in variables. Try to check this)

Given k vectors in $\mathbb{R}^n$, arrange them in matrix as usual:

$$n \text{ rows} \left\{ \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix} \right.$$

How can we produce scalar-valued antisymmetric, multilinear function?

<u>Ans.</u>: Select k rows and take determinant.

e.g. $k = 3$, $n = 7$ could select any three rows in $1, 2, \dots, 7$

We have funny notation for this which will be clearer later. For now, just use it. $dx_1 \wedge dx_4 \wedge dx_5$: pick rows $1, 4, 5$. Order matters.

So 4, 1, 5 with notation $dx_4 \wedge dx_1 \wedge dx_5 = -dx_1 \wedge dx_4 \wedge dx_5$.

(meaning select row 4 first, etc.)

e.g.
$$\begin{bmatrix} 3 & 5 & 6 \\ 1 & -2 & -1 \\ 0 & 0 & 3 \\ 2 & 0 & 0 \\ 4 & 1 & 2 \\ 0 & 2 & 1 \\ 7 & 3 & 0 \end{bmatrix}$$
v_1, v_2, v_3

then $dx_1 \wedge dx_4 \wedge dx_5 (v_1, v_2, v_3)$

$$= \det \begin{bmatrix} 3 & 5 & 6 \\ 2 & 0 & 0 \\ 4 & 1 & 2 \end{bmatrix}$$

$$= -5 \cdot 4 + 6 \cdot 2 = \boxed{-8}$$

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call them $i_1, \ldots, i_k$.

What have we done geometrically?

Selecting k rows means taking a projection from $\mathbb{R}^n \to \mathbb{R}^k$

where subspace $\mathbb{R}^k$ is spanned by basis $e_{i_1}, \ldots, e_{i_k}$

Then compute area of the k-parallelogram obtained in projection.

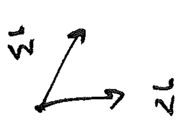
e.g. $k=2$, $n=3$ $\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{bmatrix}$ mapped by $dx_1 \wedge dx_2$ to $\det \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$

area of //-ogram in x_1-x_2 plane

Even better, it is the signed area of parallelogram.

1-forms : signed length (degenerate parallelogram)

For parallelograms, get positive answer if $\vec{v}$ lies clockwise from $\vec{w}$.



(remember $\vec{e}_1$ clockwise from $\vec{e}_2$ in standard basis

$\longleftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

What is the structure of the set of k-forms? Are we missing some?

No. : refer to forms with $dx_{i_1} \wedge \dots \wedge dx_{i_k}$ with $i_1 < i_2 < \dots < i_k$ as "elementary k-forms". We can add and scalar multiply k forms and get another k-form. (i.e. - they form a vector space.)

claim : elementary k-forms are a basis for space of k-forms on $\mathbb{R}^n$.

Even stronger : Given k-form, ϕ, we can write

rather, more specific.

$$\phi = \sum_{i_1 < \dots < i_k} \underbrace{a_{i_1, \dots, i_k}}_{= \phi(e_{i_1}, \dots, e_{i_k})} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

Hence dimension of space of k-forms in $\mathbb{R}^n$ is "n choose k"

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$