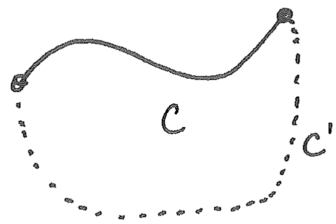
Even if want to integrate over (k-1)-manifold not a boundary, still play tricks: e.g. want to integrate over C, complete it to boundary of 2-manifold using C', where C' simple enough.



$C \cup C' = \partial X$

X: "good" 2-manifold

Stokes' theorem gives $\displaystyle\int_{\partial X} \varphi = \int_X d\varphi$

So $\displaystyle\int_C \varphi = \int_X d\varphi - \int_{C'} \varphi.$

For example, if $d\varphi = 0$ then $\displaystyle\int_C \varphi = -\underbrace{\int_{C'}}_{} \varphi$

C' any curve with same endpoints. (C' nice enough)

e.g. $\varphi = x\,dy + y\,dx$

then $d\varphi = dx \wedge dy + dy \wedge dx = 0.$

(pick 0-form f, then df is 1-form and $d(df) = 0$)

$f = xy$ then $df = dx \cdot y + dy\, x.$

Earlier sketch of Stokes' theorem:

$$\int_X d\varphi \approx \sum_{i=1}^{N} \underbrace{d\varphi(P_i)}_{} \approx \sum_{i=1}^{N} \int_{\partial P_i} \varphi \approx \int_{\partial X} \varphi$$

assume φ constant on small P_i

definition of d as flux.

orientations on boundaries cancel.

Works nicely if boundary of X is well-approximated by dyadic paving.

Easy pf of Stokes' (rather: easy case)

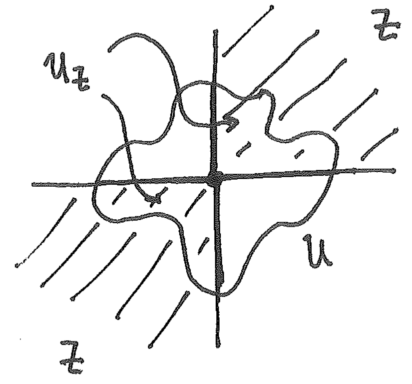
φ: $k-1$ form vanishing off of bounded open set $\mathcal{U}$.

Z: union of "quadrants" in $\mathbb{R}^k$ (regions on which coordinates $x_1, \dots, x_k$ have same sign)

$\mathcal{U}_Z = \mathcal{U} \cap Z$

then Stokes' theorem is true on $\mathcal{U}_Z$:

$$\int_{\partial \mathcal{U}_Z} \varphi = \int_{\mathcal{U}_Z} d\varphi.$$



Vanishing of φ off of $\mathcal{U}$ allows us to ignore the part of $\partial \mathcal{U}_Z$ properly contained in Z.

(put restrictions on domain $X = \mathcal{U}_Z$ so that dyadic paving is "perfect" with respect to boundary)

Idea: Argue that first two "$\approx$" in sketch of Stokes' theorem are good approximations, assumptions imply no difficulty with third "$\approx$".

So prove $\displaystyle\int_X d\varphi - \int_{\partial X} \varphi$ small ($< \epsilon$ for any $\epsilon > 0$.) by showing

$$\int_X d\varphi - \underbrace{\sum_{i=1}^{N} d\varphi(P_i)}_{\text{really mean: } P_i \text{ cubes in dyadic paving}} \text{ small}, \qquad \sum_{i=1}^{N} d\varphi(P_i) - \sum_{i=1}^{N} \int_{\partial P_i} \varphi \text{ small}$$

$$\sum_{C \in \mathcal{D}_N,\, C \subset Z} d\varphi(C)$$

first inequality ($< \epsilon$ for N suff. large) is just definition of integral as Riemann sum.

Now show: $\left| \sum_{\substack{C \in D_N \\ C \subset Z \cap U}} d\varphi(C) - \sum_{\substack{C \in D_N \\ C \subset Z \cap U}} \int_{\partial C} \varphi \right|$ small.

Analyze one cube at a time, show error small. Total # of cubes $= 2^{kN}$ covering unit k-cube in $\mathbb{R}^k$, since side length is 2^{-N}

If U bounded, then $\exists$ finite # L s.t.

# of cubes $\leq L \cdot 2^{kN}$ (choose $L = r^k$ s.t. U is in ball of radius r)

Key: indep. of N.

We need

$$\left| d\varphi\big(\underbrace{P_{\underline{x}}}_{\substack{\text{parallelogram at } x \\ = \text{ dyadic cube}}}(2^{-N}\underline{e}_1, \dots, 2^{-N}\underline{e}_k)\big) - \int_{\partial P_x(2^{-N}\underline{e}_1,\dots,2^{-N}\underline{e}_k)} \varphi \right| \leq M \cdot 2^{-N(k+1)} \quad (*)$$

then total over all cubes $\leq L \cdot 2^{kN} \cdot M \cdot 2^{-N(k+1)} = LM\, 2^{-N} \to 0$ as $N \to \infty$.

But $(*)$ is just version of definition of $d\varphi$ as flux. $h = 2^{-N}$,

$$d\varphi(P_{\underline{x}}(h\underline{e}_1, \dots, h\underline{e}_k)) = h^k\, d\varphi(P_{\underline{x}}(\underline{e}_1, \dots, \underline{e}_k))$$

$$\int_{\partial P_{\underline{x}}(h\underline{e}_1,\dots,h\underline{e}_k)} \varphi = \frac{h^k}{h^k} \int_{\partial P_{\underline{x}}(h\underline{e}_1,\dots,h\underline{e}_k)} \varphi$$

so this is

$$\underbrace{|h^k|}_{\text{know how fast it} \to 0 \text{ as } h \to 0} \cdot \underbrace{\left| d\varphi(P_{\underline{x}}(h\underline{e}_1, \dots, h\underline{e}_k)) - \frac{1}{h^k} \int_{\partial P_x(h\underline{e}_1,\dots,h\underline{e}_k)} \varphi \right|}_{\to 0 \text{ as } h \to 0 \text{ (definition of } d)}$$

There was a third "$\approx$" in our rough proof of Stokes' theorem

But with our assumptions that φ vanishes off $\mathcal{U}$, only boundary is along coordinate axes (or coordinate k-planes) — precisely the set that is always aligned with our dyadic paving.

So in our case, third "$\approx$" is actually an equality! All internal edges cancel from opposite orientations and so get exact match.

Now write:

$$\left| \int_X d\varphi - \int_{\partial X} \varphi \right| \underset{\Delta\text{-ineq.}}{\leq} \left| \int_X d\varphi - \sum_{C \in \mathcal{D}_N} d\varphi(C) \right| + \left| \sum_{C \in \mathcal{D}_N} d\varphi(C) - \sum_{C \in \mathcal{D}_N} \int_{\partial C} \varphi \right|$$

so choose $N \gg 0$ to make both pieces $< \epsilon/2$.

and we win.

What about general pf? Try to reduce to case above. ↙ or at least arbitrarily small.

cut out non-smooth boundary (with (∞-ly small) "ice cream scoop" according to H-H)

points that remain have nice nbhds. Pick point $x \in X \setminus$ scoops

Has nbhd V and diffeomorphism $F : V \to \mathbb{R}^n$ Figure A-26.2

s.t. $F(V \cap X \setminus \text{scoops}) = F(V) \cap \mathbb{R}^k \cap Z$

Z: union of quadrants

use compactness to say only need finitely many such V.