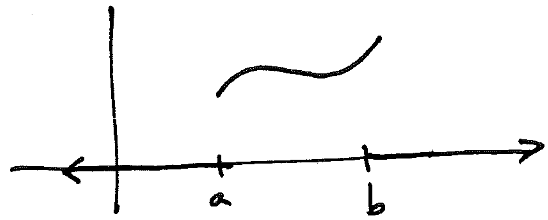
Back to theory : Which functions are integrable (using upper/lower Riemann sums in dyadic paving) ?

Small pickle: When we wanted to integrate over domain A, moved characteristic function of A into integrand, so function to be integrated might look like:



For us, satisfactory result might go as follows – <u>Theorem</u>. Let f be bounded function $\mathbb{R}^n \to \mathbb{R}$ with bounded support. Then f is integrable if f is continuous except possibly on a set of volume 0.

In above example, f is not continuous at two points a, b with volume 0.

Build our way up to this result step by step. Always assume f is bounded with bounded support

(still not enough, since $\mathbb{1}_{\text{irrationals in } [0,1]}$ not integrable)

To be integrable means that upper and lower Riemann sums are equal. Look at cube in paving $C_{k,N}$

Consider $\sup(f)$, $\inf(f)$ on $C_{k,N}$. Want this difference to be small.

<u>Rather</u>. want $\underbrace{(\sup(f) - \inf(f))}_{\overset{!!}{\operatorname{osc}_{C_{k,N}}(f)}} \cdot \operatorname{vol}(C_{k,N})$ to be small.

"oscillation of f"

But f is bounded, so $\sup_{\mathbb{R}^n}(f)$ is finite, $\inf_{\mathbb{R}^n}(f)$ is finite i.e. $\operatorname{osc}_{\mathbb{R}^n}(f)$ finite.

So plan: For any $\epsilon > 0$

$$\underbrace{U_N(f)}_{\text{upper sum at level } N} - \underbrace{L_N(f)}_{\text{lower sum at level } N} = \sum_{\substack{\text{cubes with} \\ \mathrm{osc}_{C_{k,N}}(f) > \epsilon}} \mathrm{osc}_{C_{k,N}}(f) \cdot (\mathrm{vol}\ C_{k,N}) + \underbrace{\sum_{\substack{\text{cubes with} \\ \mathrm{osc}_{C_{k,N}}(f) \le \epsilon}} \mathrm{osc}_{C_{k,N}}(f) \cdot (\mathrm{vol}\ C_{k,N})}$$

$$\le \mathrm{osc}_{\mathbb{R}^n}(f) \cdot \underbrace{\sum_{\substack{\text{cubes} \\ \text{with } \mathrm{osc}_{C_{k,N}}(f) > \epsilon}} \left(\frac{1}{2^N}\right)^n}$$

need to show this can be made arb. small, for $N \gg 0$.

$\le \epsilon \cdot$ volume of giant cube containing $\underbrace{\text{supp}(f)}$.

just needed something paralable here.

Theorem: f bounded with bounded support.

Suppose that for every $\epsilon > 0$, $\exists$ N s.t.

$$(*) \qquad \sum_{\substack{\text{cubes with} \\ \mathrm{osc}_{C_{k,N}}(f) > \epsilon}} \mathrm{vol}(C_{k,N}) < \epsilon.$$

then f integrable.

(converse also easy to see by pf. by contradiction)

Proposition: If f is uniformly continuous, f satisfies $(*)$.

remember uniform continuity means given $\epsilon > 0$, find $\delta > 0$ (independent of x_0)

s.t. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$.

In present language, given any $\epsilon > 0$, $\exists$ $\delta > 0$ s.t. $\mathrm{osc}_{B_\delta(x_0)} f < \epsilon$ $\forall x_0 \in \mathbb{R}^n$.

So if $N \gg 0$, cubes ~~in~~ $C_{K,N}$ fit into ball with diameter δ.

So $\operatorname{osc}_{C_{K,N}} f < \epsilon$ for <u>all</u> cubes of level N.

$$\Rightarrow \sum_{\substack{\text{cubes} \\ \text{with } \operatorname{osc}_{C_{K,N}}(f) > \epsilon}} \operatorname{Vol}(C_{K,N}) = 0 \quad \text{since sum is } \underline{\text{empty}}! \quad \text{so we win.}$$

Recall, a continuous function on a compact set is uniformly continuous, ~~so~~ which applies to our situation since support is bounded, so contained in compact set.

(pf. by contradiction. Good exercise in logical qualifiers)

Almost there. Just want added flexibility to remove set of volume 0 where function fails to be continuous.

<u>Plan</u>: Apply our earlier "iff" condition for integrability.

on continuous part, use supp. contained in compact set to show $\operatorname{osc}_{C_{K,N}}(f) < \epsilon$ for any given $\epsilon > 0$ if $N \gg 0$.

then on discontinuous part, where we can't control $\operatorname{osc}(f)$, show it can be fenced in by cubes whose total volume is $\leq \epsilon$.

Proof would be easy except that it is possible that point of discontinuity could lie on boundary of cube, and thus be a limit point of points ~~[illegible]~~ from continuous region of support. So have to make "buffer zone". Any cube can be completely surrounded using $3^n - 1$ cubes. (Rubik's cube minus center cube we're surrounding)