Numeric integration in 1-variable:

Discussed this briefly last semester when discussing interpolation of polynomials. Idea: Given function f, model it by quadratic passing through three equally spaced points

$T: P_{\le 2} \to \mathbb{R}^3$ found matrix for T, inverted

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = p \longmapsto \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ \frac{1}{2}(y_2 - y_0) \\ \frac{1}{2}(y_0 - 2y_1 + y_2) \end{bmatrix} \longleftarrow\!\shortmid \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

integrate from -1 to 1 ($p(x)$ is our approx. of $f(x)$)

to get $\underbrace{\frac{1}{3}}(y_0 + 4y_1 + y_2)$

$\frac{2}{6}$, special case of $\frac{b-a}{6}$

Leads to general formula:

$$\int_a^b f(x) \approx \frac{b-a}{6n}\Big(f(x_0) + 4f(x_1) + \underbrace{f(x_2) + f(x_2)}_{2f(x_2)} + 4f(x_3) + \cdots + f(x_{2n})\Big)$$

Surprise: models cubic functions perfectly (reason it is better than quadratic: $\int_{-1}^1 x^3\,dx = 0$.)

So error

$$\int_a^b f(x) - \text{Simpson's rule for } f = \frac{(b-a)^5}{2880\,n^4} f^{(4)}(c) \text{ for some } c \in (a,b).$$

Also discussed Bernstein polynomials $x^{n-k}(1-x)^k$ $k = 0, \ldots, n$ instead of monomials.

Book also discusses Gaussian integration:

Again p of degree $\le d$. Pick points $x_1, \dots, x_m$ and weightings $w_1, \dots, w_m$ so that, for all p of deg $\le d$ (idea: want m small)

$$\int_{-1}^{1} p(x)\,dx = \sum_{i=1}^{m} w_i\, p(x_i) \qquad \left(\text{Simpson: } \begin{array}{l} x_i = -1, 0, 1 \\ w_i = 1, 4, 1 \end{array} \right)$$

$2m$ unknowns. Solve $d+1$ equations, one for $1, x, \dots, x^d$.

Try to solve with $2m \geq d+1$. If $d = 3$, can try with $m = 2$ (beat Simpson, using $m=3$)

Get $$\int_{-1}^{1} 1\,dx = w_1 + w_2 = 2$$

$$\int_{-1}^{1} x\,dx = w_1 x_1 + w_2 x_2 = 0$$

$$\int_{-1}^{1} x^2\,dx = w_1 x_1^2 + w_2 x_2^2 = \frac{2}{3}$$

$$\int_{-1}^{1} x^3\,dx = w_1 x_1^3 + w_2 x_2^3 = 0$$

Bad because non-linear equations. Make some assumptions: $x_1 = x$, $x_2 = -x$, $w_1 = w_2 = w$.

then eqns 2+4 are true. Left with

$$2w = 2, \quad 2wx^2 = \frac{2}{3} \quad \text{so} \quad w = 1 \quad x = \sqrt{\frac{1}{3}}.$$

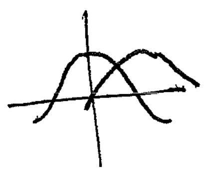
Example: $\displaystyle\int_{-1}^{1} \cos x\,dx = \sin x \Big|_{-1}^{1} = \sin(+1) - \sin(-1)$

$= 2\sin 1 \approx 1.6829..$

Simpson: $\approx \frac{1}{3}\left(\cos(-1) + 4\cos(0) + \cos(1) \right) = \frac{4}{3} + \frac{2}{3}\cos(1) \approx 1.6935$

Gaussian: $\approx \cos\left(1/\sqrt{3}\right) + \cos\left(-1/\sqrt{3}\right) = 2\cos\left(1/\sqrt{3}\right) \approx 1.6758$

(also recall



$\cos(x - \pi/2) = \sin x$
$\quad\shortparallel$
$\cos(\pi/2 - x)$

$\pi/2 - 1/\sqrt{3} = .9934...$

Try to similarly handle higher degree d by setting $x_i = -x_{2m-i+1}$, eliminates all odd powered equations.

In practice, also use Gaussian rules for integrations like

$$\int_{-\infty}^{\infty} f(x) \frac{e^{-x^2}}{\sqrt{\pi}}\,dx$$, where we find answers for f of small degree as before.

Useful in probability. Only compute w_i, x_i once and then can use on all choices of f.

In any case, all rules associate weights w_i and approximate values

for: $\displaystyle\sum_{i=1}^{k} w_i f(p_i)$ $\quad p_i$: distinguished points in $[a,b]$.

to $\displaystyle\int_a^b f(x)\,dx$

Not Fubini.
More Basic: §4.1

In $\mathbb{R}^2$: $\displaystyle\int_{[a,b]^2} f(\underline{x})\,|d^2x| = \int_{[a,b]} f_1(x_1)\,dx_1 \int_{[a,b]} f_2(x_2)\,dx_2$

$\displaystyle\sum_i w_i f_1(p_i) \qquad \sum_i w_i f_2(p_i)$

$\quad\approx \qquad\qquad \approx$

if $f(\underline{x}) = f_1(x_1) f_2(x_2)$

$$= \sum_{1 \le i_1, i_2 \le k} w_{i_1} w_{i_2} \underbrace{f_1(p_{i_1}) f_2(p_{i_2})}_{f\begin{pmatrix} p_{i_1} \\ p_{i_2} \end{pmatrix}} .$$

So we can handle functions of several variables. Problem is that when dimension increases, sampling grows exponentially. Simpson, with 10 points a side for cube in dimension 8 gives 10^8 computations of special values of function. (ok for fast computer. Any bigger we need another way!)

<u>Fix</u>: Sample points randomly in high dimensional cube and ~~take~~ average.

$$\text{Know} \quad \int_A f(\underline{x}) \, |d^n \underline{x}| = \begin{pmatrix} \text{average val. of} \\ f \text{ on } A \end{pmatrix} \cdot (\text{vol}(A))$$

So sampling and averaging evaluates: $\displaystyle \int_A f(\underline{x}) \, |d^n \underline{x}| \Big/ \text{vol}(A).$

(if don't know volume of A in advance put A in a box B and do Monte Carlo method for

$$\int_B 1_A \, |d^n x| \Big/ \text{vol}(B).)$$

<u>Problem</u>: How do we know if we're close to the right answer?

<u>Ans</u>: Approximate the standard deviation and use central limit theorem.