Last week, discussed numeric integration in 1-variable.

Always of form $\sum_{i=1}^{m} w_i f(x_i)$. Chosen to match polys. f of degree $\le d$ exactly.

w_i: weight; x_i: sample points.

In several variables, if f factors by component:

$$f(\underline{x}) = f_1(x_1) \cdots f_n(x_n) \quad \text{for some functions } f_1, \ldots, f_n,$$

use same numeric integration.

Easy result: if w_i, x_i give exact answers for one-var. f of deg $\le d$

then
$$\int_{\mathbb{R}^n} f(\underline{x})\, |d^n x| = \left(\sum_i w_i f_1(x_i)\right) \cdots \left(\sum_i w_i f_n(x_i)\right)$$
for all $f = f_1 \cdots f_n$ with f_i of deg $\le d$.

Use this to do multi-variate version of our weighting rule. Reason: monomials do factor.

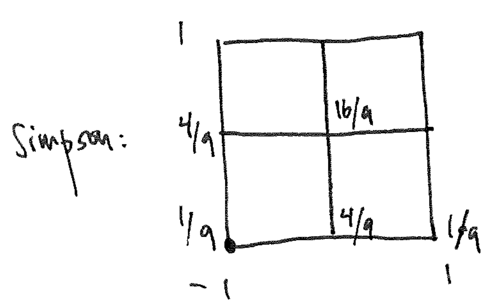
So 2-var. Simpson is exact on $x^3 y^3$ or $x^2 y$ or ...

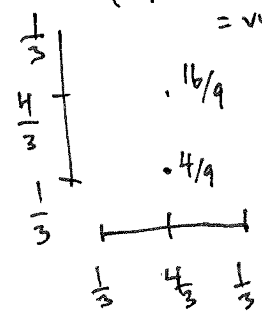
So match 2-var. Taylor poly at all terms with deg. ≤ 3 in each variable

2 dimensional weighting:

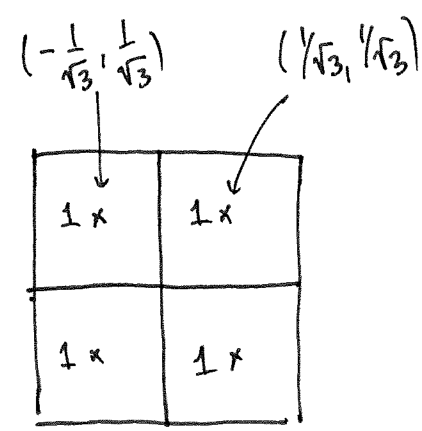
(wts. add to 4 = vol. of box)

Simpson:





Gauss:



$$= \sum_{1 \le i_1, i_2 \le k} w_{i_1} w_{i_2} \underbrace{f_1(p_{i_1}) f_2(p_{i_2})}_{f\begin{pmatrix} p_{i_1} \\ p_{i_2}\end{pmatrix}} .$$

So we can handle functions of several variables. Problem is that when dimension increases, sampling grows exponentially. Simpson, with 10 points a side for cube in dimension 8 gives 10^8 computations of special values of function. (ok for fast computer. Any bigger we need another way!)

Fix: Sample points randomly in high dimensional cube and take average.

Know $$\int_A f(\underline{x})\, |d^n \underline{x}| = \begin{pmatrix} \text{average val. of} \\ f \text{ on} \\ A \end{pmatrix} \cdot \big(\text{vol}(A)\big)$$

So sampling and averaging evaluates: $$\int_A f(\underline{x})\, |d^n \underline{x}| \Big/ \text{vol}(A).$$

(if don't know volume of A in advance put A in a box B and do Monte Carlo method for

$$\int_B 1_A\, |d^n x| \Big/ \text{vol}(B).\ )$$

Problem: How do we know if we're close to the right answer?

Ans: Approximate the standard deviation and use central limit theorem.

Approximation to standard deviation:

$$\bar{s}^2 := \frac{1}{N} \sum_{i=1}^{N} (a_i - \bar{a})^2 = \left(\frac{1}{N} \sum_{i=1}^{N} a_i^2 \right) - \bar{a}^2$$

Recall $\sigma(f) = \sqrt{Var(f)}$ with $Var(f) = E(f - E(f))^2$

$$= E(f^2) - (E(f))^2$$

where $E(f)$ = expectation of f

$$= \int_{\mathbb{R}^n} f \, d\mu(\underline{x})$$

Central limit theorem: $P(\bar{a} \in [E + a\sigma/\sqrt{N}, E + b\sigma/\sqrt{N}])$

average over N trials

is roughly $\frac{1}{\sqrt{2\pi}} \int_a^b e^{-t^2/2} \, dt$.

Allows us to solve for N based on desired accuracy.

Example : $\displaystyle\int_{\mathbb{R}^2} 1_{[-1,1]\times[-1,1]}(\underline{x}) \cdot \cos(x_1x_2)\ |d^2(x_1,x_2)|$

Simpson : $1/9 \cos(-1\cdot-1) + 4/9 \cos(0\cdot-1) + 1/9 \cos(1\cdot-1)$

$+ \cdots$

Gauss : $\cos\left(\frac{-1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\right) + \cos\left(\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\right) + \cdots$

$= 4\cos\left(\frac{1}{3}\right). \approx 3.7798\ldots$

Exact? $\displaystyle\int_{-1}^{1}\int_{-1}^{1} \cos(x_1x_2)\, dx_1\, dx_2 = \int_{-1}^{1} \frac{\sin(x_1x_2)}{x_2}\Bigg|_{-1}^{1} dx_2$

$\displaystyle= \int_{-1}^{1} \frac{2\sin x_2}{x_2}\, dx_2$

$\approx 2\times 1.89217$ (no elem. anti-deriv.)

≈ 3.78433

10 ~~Four~~ random #'s :

Trial	Ans
1	.9246...
2	.9536...
3	.9014...
4	.9492...
5	.9215...

Avg. .93.

vs. $3.78433/4 = .94608.$

More tests:

10 trials

$\bar{a}$: .94915 $\quad \bar{s}$ = .0738

.97356 $\quad$ " = .0308

.9379 $\quad$ = .0536

$\bar{a}$: approximation of E. Want accuracy : $\left| \frac{E - \bar{a}}{E} \right|$ small. e.g. $\leq .01$

with certain probability.

(depends on standard deviations under bell curve)

central limit thm :

$\bar{a} \in \left[E - .001_{E}, E + \underbrace{.001_{E}}_{b\sigma/\sqrt{N}} \right]$

2 std deviations = 95%

3 std. deviations = 99.7%

(1 std. deviation = 68.27%)

$b = 3$

Find N s.t. $3\sigma / \sqrt{N} E \leq .01$

i.e. $\frac{300\sigma}{E} \leq \sqrt{N}$

So $N \geq \left(\frac{300\sigma}{E} \right)^2 = (23.3)^2$

≈ 544

$E \sim .9$

$\sigma \sim .07 \quad (.1)?$

(if only want 1 std. dev. : $\left(\frac{100\sigma}{E} \right)^2 \approx 60$.

60 trials :

.93171

.9552

.9525

.9618

5000 :

.94655...

50,000 :

.94585...

.94598

.94628

500k :

.94621

.94592

550 trials :

.94528

.94326

.94677

5mil : .946076...