Define algebra $U_q(\mathfrak{sl}_2)$ by $\Big\langle E, F, K, K^{-1} \;\Big|\; KK^{-1} = 1 = K^{-1}K$

$$KEK^{-1} = q^2 E$$
$$KFK^{-1} = q^{-2} F$$
$$[E,F] := EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \Big\rangle$$

(Initially, ground field k with $q \neq 0, \pm 1 \in k$)

Things we'd like to know/show about $U_q(\mathfrak{sl}_2)$:

① - has a Hopf algebra structure

remember, as an algebra.

② - what are its finite dim'l modules? How do they compare to those of $SL(2,\mathbb{C})$, $\mathfrak{sl}_2(\mathbb{C})$, $U(\mathfrak{sl}_2)$?

(dichotomy if q is rt. of unity or not, already seen that q-integers $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$ or $(n)_q = \dfrac{q^n - 1}{q - 1}$ appear in formulas for generators / rel'ns, and they can be 0 if q is rt. of unity)

③ - is it quasitriangular? Is there a way to compute associated $R \in H \otimes H$? (abstract QYBE)

④ - can we see this construction of $U_q(\mathfrak{sl}_2)$ arising from some general process? Natural construction?

Hopf algebra structure is familiar to us from $U_q(\mathfrak{b}_+)$. -

<u>Moral</u>: Define on generators. Extend Δ, ε multiplicatively, S as an antimorphism.

(since Δ, ε have to be algebra maps)

Just as before, let

$$\Delta(K) = K \otimes K$$
$$\Delta(E) = E \otimes 1 + K \otimes E$$

+ <u>new rel'n</u>: $\Delta(F) = F \otimes K^{-1} + 1 \otimes F$.

co<u>unit</u>: $\varepsilon(K) = 1$

$\varepsilon(E) = \varepsilon(F) = 0$.

with Δ, ε compatible (satisfy triangle axiom)

check compatible with rel'ns, coassociative

Finally, unique antiautom S with $S(E) = -K^{-1}E$, $S(F) = -FK$, $S(K) = K^{-1}$.

claim 1: $U_q(sl_2) \simeq U_q' = \langle E, F, K, K^{-1}, L \mid KK^{-1} = K^{-1}K = 1,\ KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F,\ [E,F] = L,\ (q-q^{-1})L = K - K^{-1},\ [L,E] = q(EK + K^{-1}E),\ [L,F] = -q^{-1}(FK + K^{-1}F) \rangle$

U_q' makes sense for all q, including $q=1$.

claim 2: $U_1' \simeq U(sl_2)[K]/(K^2-1)$

so we get a projection of

$$U_1' \longrightarrow U(sl_2)$$

$$E \longrightarrow X$$
$$F \longrightarrow Y$$
$$K \longrightarrow 1$$
$$L \longrightarrow H$$

rewrite last relations since K central with $K^2=1$.

isomorphism given by $E \mapsto XK$, $F \mapsto Y$, $K \mapsto K$, $L \mapsto HK$.

pf of claim 1: $\varphi : U_q \longrightarrow U_q'$

$$E \longmapsto E$$
$$F \longmapsto F$$
$$K \longmapsto K$$

want ψ map back whose composition is the identity.

$\psi(E) = E$, $\psi(F) = F$, $\psi(K) = K$

$\psi(L) = ?$

want relations to hold in U_q (i.e. well-defined)

so set $\psi(L) = [E,F]$ since $\psi(E) = E$, $\psi(F) = F$.

Now check last 3 relations, + confirm compositions are identity by checking on generators.

e.g. $(q-q^{-1})\psi(L) = (q-q^{-1})[E,F]$

$= K - K^{-1}$. ✓

reln in $U_q(sl_2)$.

Back to rep'n thy. of $SL(2) \rightsquigarrow \mathfrak{sl}(2) \rightsquigarrow \mathcal{U}(\mathfrak{sl}(2))$ ← eventually put a new spin on this w/ Hopf algebras.

$$\mathcal{U}(\mathfrak{sl}_2) = \langle X, Y, H \mid [X,Y] = H,\ [H,X] = 2X,\ [H,Y] = -2Y \rangle$$

<u>Facts</u>: ① $X^p H^q = (H-2p)^q X^p \qquad Y^p H^q = (H+2p)^q Y^p$

(proof by induction)

② $\{X^i Y^j H^k\}_{i,j,k \in \mathbb{Z}_{\geq 0}}$ is a basis of $\mathcal{U}(\mathfrak{sl}_2)$ (similar to our proof for $\mathcal{U}_q(\mathfrak{b}_+)$)

③ $C = XY + YX + \dfrac{H^2}{2}$ is in $\mathfrak{Z} :=$ center of $\mathcal{U}(\mathfrak{sl}_2)$.

(check that Lie brackets vanish. Enough to check on generators,

For example
$$[H,C] = \underbrace{[H, XY]}_{\substack{HXY - XYH \\ = HXY - XHY + XHY - XYH \\ = [H,X]Y + X[H,Y] \\ = 2XY - 2XY = 0}} + [H, YX] + \tfrac{1}{2}\underbrace{[H,H^2]}_{=0}$$

$[H,YX]$: similar to this case.

In fact, better: $\mathfrak{Z} = \langle C \rangle$. (idea: Harish-Chandra constructed isom.

$$\mathfrak{Z}(\mathcal{U}) \xrightarrow{\sim} \Bbbk[t] \qquad \Bbbk = \mathbb{C} \text{ for simplicity}$$
$$C \longmapsto t \;)$$

Now determine all fin. dim'l $\mathcal{U}(\mathfrak{sl}_2)$ modules.

Key concept: "highest weight vector" / "highest weight rep'n"

A vector $v \neq 0 \in V$: $\mathcal{U}(\mathfrak{sl}_2)$-module is "of weight λ" ($\lambda \in \Bbbk$) if $H \cdot v = \lambda v$. and is "highest weight" if, in addition $Xv = 0$.