Last time: $\mathcal{U}_q(\mathfrak{sl}_2)$ has Hopf alg. structure. Compare its rep'ns to those of $\mathcal{U}(\mathfrak{sl}_2)$.
(and a presentation (longer) which shows $\mathcal{U}(\mathfrak{sl}_2)$ as quotient)

had $\mathcal{U}_1'$ with

$$\mathcal{U} \simeq \mathcal{U}_1'/(K-1)$$

(last time, equivalently wrote
$$\mathcal{U}_1' \simeq \mathcal{U}[K]/(K^2-1)$$
)

Recorded 3 facts about $\mathcal{U}(\mathfrak{sl}_2)$:

① $X^p H^q = (H-2p)^q X^p$

(induction, using $[H,X] = 2X$)

$Y^p H^q = (H+2p)^q Y^p$

(using $[H,Y] = -2Y$)

② $\{X^i Y^j H^k\}_{i,j,k \geq 0}$ is a basis for $\mathcal{U}(\mathfrak{sl}_2)$

③ $C := XY + YX + \frac{H^2}{2}$ is central.

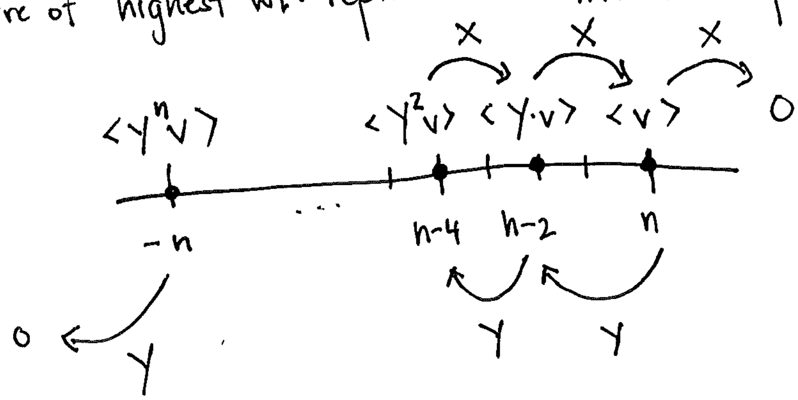
(in fact, it generates the center)

general results of Harish-Chandra, applied to $\mathcal{U}(\mathfrak{sl}_2)$

$$\begin{aligned} Z(\mathcal{U}(\mathfrak{sl}_2)) &\xrightarrow{\sim} k[t] \\ C &\longmapsto t \end{aligned}$$

showing C generates center.

"Picture" of highest wt. rep'n: $n+1$ dim'l rep'n.



Key point: $XY v_p \neq v_p$ as vectors in wt. space

(maybe talk a bit more about higher rank Lie algebras)

Back to rep'n thy. of $SL(2) \rightsquigarrow sl(2) \rightsquigarrow \mathcal{U}(sl(2))$ ⟵ eventually put a new spin on this w/ Hopf algebras.

$$\mathcal{U}(sl_2) = \langle X, Y, H \mid [X,Y] = H,\ [H,X] = 2X,\ [H,Y] = -2Y \rangle$$

Facts: ① $X^p H^q = (H-2p)^q X^p \qquad Y^p H^q = (H+2p)^q Y^p$

(proof by induction)

② $\{X^i Y^j H^k\}_{i,j,k \in \mathbb{Z}_{\geq 0}}$ is a basis of $\mathcal{U}(sl_2)$ (similar to our proof for $\mathcal{U}_q(b_+)$)

③ $C = XY + YX + \frac{H^2}{2}$ is in $\mathfrak{Z} :=$ center of $\mathcal{U}(sl_2)$.

(check that Lie brackets vanish. Enough to check on generators, ~~which~~

For example $[H,C] = \underbrace{[H,XY]}_{} + [H,YX] + \frac{1}{2}\underbrace{[H,H^2]}_{=0}$

$[H,YX]$: similar to this case.

$$\begin{aligned} [H,XY] &= HXY - XYH \\ &= HXY - XHY + XHY - XYH \\ &= \underbrace{\underbrace{[H,X]}_{2X}Y + X\underbrace{[H,Y]}_{-2Y}}_{=0}. \end{aligned}$$

In fact, better: $\mathfrak{Z} = \langle C \rangle$. (idea: Harish-Chandra constructed isom.

$$\mathfrak{Z}(\mathcal{U}) \xrightarrow{\sim} k[t] \qquad k = \mathbb{C} \text{ for simplicity}$$
$$C \longmapsto t \ )$$

Now determine all fin. dim'l $\mathcal{U}(sl_2)$ modules.

Key concept: "highest weight vector" / "highest weight rep'n"

A vector $v \neq 0 \in V$: $\mathcal{U}(sl_2)$-module is "of weight λ" ($\lambda \in k$) if

$H \cdot v = \lambda v$. and is "highest weight" if, in addition $Xv = 0$.

Proposition: Every non-zero f-dim'l $\mathcal{U}(\mathfrak{sl}_2)$-module V has highest wt. vector.

pf: Over $\mathbb{C}$, H has eigenvector $w \neq 0$ with e-value α.

if $Xw = 0$, then done. If not consider $\{X^n w\}_{n \geq 0}$

then $H(X^n w) = (HX^n)\cdot(w) \underset{\text{fact ①}}{=} (X^n H + 2n X^n)\, w$

$= (\alpha + 2n) X^n w$

So $\{X^n w\}$ has distinct e-values $\{\alpha + 2n\}$.

But if V is finite dim'l, then only finite # of evalues

so must have some n s.t. $X^n w \neq 0$, $X^{n+1} w = 0$ (and hence $X^k w = 0 \ \forall k \geq n+1$)

then $X^n w$ is the highest weight vector.

Thm: Simple modules of $\mathcal{U}(\mathfrak{sl}_2)$ are (up to isom.) indexed by non-negative integers n, call them $V(n)$, of dimension $n+1$, with h.w. vector of wt. n. (wt. vectors have wts: $n, n-2, n-4, \dots, -n$)

and necessarily form basis of $V(n)$

X raises wt. by 2, Y action lowers wt. by 2.

and Casimir elt. acts as scalar on $V(n)$. scalar is $\dfrac{n\cdot(n+2)}{2}$.

this last fact may be used to prove that any fin. dim'l $\mathcal{U}(\mathfrak{sl}_2)$-module is "semisimple" — able to be decomposed into direct sum of simple modules. (construct op. acting as identity on simple submodules)

✓ Have $\{H_\alpha, X_\alpha, Y_\alpha\}_{\alpha \in \Phi^+}$

(rep'n theory of $\mathcal{U}(\mathfrak{g})$ not much harder. Vector-valued version of this story.)

Express this basis in terms of highest weight vector v:

Let $v_p := \frac{1}{p!} Y^p v \qquad p = 0, \dots, \lambda$.

Basic question - Given $V(n), V(m)$. Form a new module $V(n) \otimes V(m)$ using bialgebra structure. How does it decompose into irreducibles? (simple modules)

Clebsch-Gordan formula: $V(n) \otimes V(m) \cong V(n+m) \oplus V(n+m-2) \oplus \cdots \oplus V(n-m)$

pf: Quickly confirm that dimensions of both sides match $= (n+1)(m+1)$.

So done if we can find h.w. vectors for weight $n+m-2p \quad p \in [0, m]$ in $V(n) \otimes V(m)$. (b/c h.w. vector generates simple module $V(n+m-2p)$ i.e. giving an embedding into $V(n) \otimes V(m)$, with no kernel and since highest weights distinct, their sums are direct.

check that if

$v_p := \frac{1}{p!} Y^p v \quad v$: hw for $V(n)$

$v'_p := \frac{1}{p!} Y^p v' \quad v'$: hw for $V(m)$

then given $p \in [0, m]$,

$$\sum_{i=0}^{p} \frac{(m-p+i)!\,(n-i)!}{(m-p)!\,n!} v_i \otimes v'_{p-i}$$

is the desired h.w. vector for $V(n+m-2p)$.

(tensors all have right wt. Just need to solve for coeffs.)

Introduce Hopf alg. point of view:

Def'n: Given H: bialgebra, A: algebra, A is a module-algebra $/H$ if

- as a vector space, A has H-module structure.
- $\mu : A \otimes A \to A$, $\eta : k \to A$ are H-module maps (comult and counit give H action on $A \otimes A$, k)