LEC 18 INTRO

Last week, made a lot of progress on rep'n thy of $U_q := U_q(\mathfrak{sl}_2)$.

~~When~~ When k any field with $\mathrm{char}(k) \neq 2$, $q \in k^\times$ not a root of unity.

then any fin. dim'l module is a sum of weight spaces (eigenspaces under group-like K) with weights $\pm q^a$, $a \in \mathbb{Z}$.

Price we paid for working w/ k not nec. alg. closed:

Have hierarchy for any given matrix $T \in \mathrm{End}(V)$:

- min poly. of T splits into distinct linear factors $\longleftrightarrow$ diagonalizable, V direct sum of eigenspaces.
- char. poly of T splits into linear factors $\longleftrightarrow$ Jordan canonical form.

with cartoon expression
$$\begin{pmatrix} \boxed{J_1} & & \\ & \ddots & \\ & & \boxed{J_k} \end{pmatrix} \qquad J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$$

with basis vectors $v, (T-\lambda_i I)v, \ldots (T-\lambda_i I)^{p-1} v$ where p is size of block.

"generalized eigenvectors"

V is direct sum of gen'd e-vectors $\{ v \mid (T-\lambda_i I)^s \cdot v = 0,\ s \gg 0 \}$

- In general, char poly of T expressible as products of irreducible polys. with multiplicity.

If $\phi_i(x)$ is one such irred. factor,
$$V(\phi_i) = \{ v \in V \mid \phi_i(T)^s \cdot v = 0,\ s \gg 0 \}$$

then "rational canonical form" in which
$$\begin{pmatrix} 0 & & & -a_0 \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & \vdots \\ & & 1 & -a_{p-1} \end{pmatrix}$$
have char poly. $\phi_i(x)^{m_i}$

again sums are direct, since polys. are irreducible.

(Reference: Friedberg, Insell, Spence - very last ~~pages~~ page Thm. 7.24.)

mid March. $\rightsquigarrow$ ch. 12 in Dummit-Foote.

And hence the minimal polynomial of K divides the associated polynomial to the above (replace K with variable x).

So we're done! Because this polynomial splits into linear factors with distinct roots, so the min. poly. must also.

(if you were worrying that maybe $-q^a = q^b$ some a, b, note this implies $q^{b-a} = -1$ so $q^{2(b-a)} = 1$ ↯ q not a root of unity.)

— END OF LECTURE 17 ——

Now we know that fin-dim'l modules are entirely comprised of wt. spaces.

let's explore highest weights: $m \in M$, $Em = 0$, $Km = \lambda m$.

There's a natural "universal" module with this property:

$$M(\lambda) := \mathcal{U}_q / \mathcal{U}_q E + \mathcal{U}_q (K - \lambda) .$$

Any M with h.w. vector m of wt. λ has a unique homom. $M(\lambda) \longrightarrow M$

$[\cdot] \to [1] \longrightarrow m$

for coset rep classes.

This $M(\lambda)$ has basis $[F^i]_{i \geq 0}$

by PBW theorem, call these $m_0, m_1, m_2, \ldots$

then our relations imply:

(infinite dim'l module)

$$F \cdot m_i = m_{i+1}$$

$$(**) \qquad E \cdot m_i = \begin{cases} 0 & \text{if } i = 0 \\ [i]_q \dfrac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q - q^{-1}} m_{i-1} & \text{otherwise.} \end{cases}$$

$$K \cdot m_i = \lambda \cdot q^{-2i} m_i$$

⟵ already make some observations about λ being integer power of q or no

Pretty close... Now every fin. dim'l module $M = \bigoplus_\mu M_\mu$, so we can find some λ with $M_\lambda \neq 0$, $M_{q^2\lambda} = 0$. (Remember this is the right criterion b/c $E \cdot M_\lambda \subseteq M_{q^2\lambda}$, so $m \in M_\lambda$ is a highest wt. vector)

Then there is a non-zero homom. from universal module

$$\varphi: M(\lambda) \longrightarrow M.$$

If M is simple, this is surjective, and we have exact sequence

$$0 \to \operatorname{Ker}\varphi \to M(\lambda) \twoheadrightarrow M \to 0$$

with $M \simeq M(\lambda)/\operatorname{Ker}\varphi$.

So it remains to study when universal module $M(\lambda)$ has submodules!

Proposition: If q not a rt. of 1. $\lambda \in k$, $\neq 0$.

if $\lambda \neq \pm q^n$ for any $n \geq 0$, universal module $M(\lambda)$ is simple.

if $\lambda = \pm q^n$, some $n \geq 0$, then $\langle \{m_i\}_{i \geq n+1} \rangle$ is the unique non-trivial submodule of $M(\lambda)$, isomorphic to $M(q^{-2(n+1)}\lambda)$.

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pf: Suppose M' is a submodule of $M(\lambda)$. Then, as a U_q-module, M' is a direct sum of its weight spaces. Know that $M(\lambda)$ has one dim'l weight spaces $m_i := [F^i]$ with wt $q^{-2i}\lambda$. (using that q not a root of 1 to guarantee distinct evalues.)

So M' is spanned by a subset of these m_i.

Pick $j \geq 0$ minimal among these. ($m_0 = [1]$ is highest wt. λ for $M(\lambda)$)

M' is closed under action of F, so M' contains precisely those m_i with $i \geq j$.

if $j=0$, then $M' = M(\lambda)$, so assume $j>0$. then $E \cdot m_j$ is a const. multiple of m_{j-1} ($\notin M'$, since we chose j minimal). But M' closed under E so must be that $E \cdot m_j = 0$. Returning to our identities in $M(\lambda)$, this can only result from $\lambda q^{1-j} - \lambda^{-1} q^{j-1} = 0$, so $\lambda^2 = q^{2(j-1)}$ i.e. $\lambda = \pm q^{(j-1)}$.

we know how generators act, and basis for purported module.

So we've proved that $M(\lambda)$ is simple whenever $\lambda \neq \pm q^n$ and if $\lambda = \pm q^n$, there is at most one proper submodule. It has $E \cdot m_{n+1} = 0$. Remember m_{n+1} has wt. $q^{-2(n+1)}\lambda = \pm q^{-n-2}$.

By the universal property, there is a homom.

$$M(\pm q^{-n-2}) \longrightarrow M(\lambda)$$
$$m_0 \longmapsto m_{n+1}$$

this module is simple as its exponent of q is negative ← so this map must be an isomorphism.

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this completes classification — For every $n \geq 0$, there are a pair of simple U_q-modules $L(n,+)$ with basis $m_0, \dots, m_n$, m_0 of wt. q^n
$L(n,-)$ with basis $m_0', \dots, m_n'$, m_0' of wt. $-q^n$

where $L(n,\pm) \simeq M(\pm q^n)/M'$ (M': unique submod. spanned by m_i, $i>n$)

Left to do: fin. dim'l modules are semisimple. Then on to root of unity case...