The case when q is a rt. of 1. So assume that has order ℓ : $q^\ell = 1$.

this implies $[\ell]_q = 0$ and so $[i]_q! = 0$ when $i \geq \ell > 2$.

Proposition : $E^\ell, F^\ell, K^\ell, K^{-\ell}$ are in center of $\mathcal{U}_q$. (check on generators)

pf : repeated use of $KEK^{-1} = q^2 E$, $KFK^{-1} = q^{-2} F$ shows

$K^\ell, K^{-\ell}$ commutes with E, F, so $K^\ell, K^{-\ell}$ are central.

also $KE^\ell K^{-1} = q^{2\ell} E^\ell$, similarly for F. Finally $[E, F^s] = [s]_q F^{s-1} \cdot$ binomial (expression in K)

if $s = \ell$, this is 0.

Similarly for $[F, E^s]$. //

in fact, closer look reveals if ℓ even, then with $\ell = 2\ell'$

in fact $[\ell']_q = 0$ since numerator is $q^{\ell'} - q^{-\ell'}$; which shows

that $E^{\ell'}, F^{\ell'}, K^{\ell'}, K^{-\ell'}$ are central.

From now on, we assume that ℓ is odd, $\ell \geq 3$. (if ℓ even, just replace statements with ℓ' s.t. $\ell = 2\ell'$)

Return to our universal module $M(\lambda)$, $\lambda \in k^\times$.

then $K \cdot m_\ell = \lambda \cdot q^{-2\ell} m_\ell = \lambda m_\ell$.

$E \cdot m_\ell = 0$ since its action involved $[\ell]_q$.

so any vector of form $m_\ell - b m_0$, any $b \in k$, is a highest weight vector in $M(\lambda)$.

Consider : $Z_b(\lambda) := M(\lambda) / \mathcal{U} \cdot (m_\ell - b m_0)$

$\mathcal{U}(m_\ell - bm_0)$ is spanned by $F^i(m_\ell - bm_0) = m_{i+\ell} - bm_i$, $i \geq 0$,

since $K(m_\ell - bm_0) = \lambda(m_\ell - bm_0)$, $E(m_\ell - bm_0) = 0$.

Thus $Z_b(\lambda)$ ~~is spanned by~~ has basis given by $\{m_j\}_{j<\ell}$ (or rather $[m_j]$, a little silly since $[[F^j]]$)

(ℓ-dimensional module) with

$$K \cdot m_i = \lambda q^{-2i} m_i \,, \quad F \cdot m_i = \begin{cases} m_{i+1} & \text{if } i < \ell-1 \\ b \cdot m_0 & \text{if } i = \ell-1 \end{cases} \,,$$

$$E \cdot m_i = \begin{cases} 0 & i=0 \\ [i]_q \dfrac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q-q^{-1}} m_{i-1} & i>0 \,. \end{cases}$$

and since $\lambda \cdot q^{-2i}$, $0 \le i < \ell$ are distinct, wt spaces are all 1-dim'l gen'd by m_j, $j<\ell$.

Note: $F^\ell \cdot m_0 = b m_0$, so if $b \neq 0$, then

- F doesn't act nilpotently on $Z_b(\lambda)$ (unlike before, when q not rt of 1, $F^s M = 0$ $s \gg 0$)
- λ need not be of the form $\pm q^a$, $a \in \mathbb{Z}$.

Is this a simple module?

Prop. (2.12 in Jantzen) q: primitive ℓ-th rt of 1, ℓ odd, ≥ 3.

if $b \neq 0$ or if $\lambda^{2\ell} \neq 1$ then $Z_b(\lambda)$ is a simple $\mathcal{U}_q$-module.

if $b=0$ and $\lambda = \pm q^n$ $(0 \le n < \ell)$ then $Z_b(\lambda)$ is:

simple if $n = \ell-1$.

if $n < \ell-1$, then $\langle v_j \mid j > n \rangle$ span a submodule of $Z_b(\lambda)$ and this is unique non-trivial submod.

pf: Let M be non-zero submodule. $Z_b(\lambda)$ is a direct sum of its one dim'l wt. spaces and M is closed under action of K, so M is direct sum of weight spaces $M \cap Z_b(\lambda)_\mu$. Weight vectors are those $m_i \in M$. our named basis above.

choose $j \geq 0$ minimal with $m_j \in M$. M is closed under F, so

$m_i = F^{i-j} m_j \in M$ for all $j \leq i < \ell$. As in our earlier pf, if $j = 0$ then $M \cong Z_b(\lambda)$, i.e. not proper.

So we may assume $j > 0$.

Now the weird property of F acting by $F \cdot m_{\ell-1} = b m_0$ says this only happens if $b = 0$ (necessary condition for proper submodule). With $b = 0$:

M spanned by m_i, $i \geq j$, and $E \cdot m_j$ is a multiple of m_{j-1}, so by minimality of j, and M closed under E, must be $E \cdot m_j = 0$.

but then knowing E acts in layer $Z_b(\lambda)$ by $[j]_q \dfrac{\lambda q^{1-j} - \lambda^{-1} q^{j-1}}{q - q^{-1}} \cdot m_{j-1}$

on m_j, then numerator is 0. Can't be $[j]_q$ giving 0, since $0 < j < \ell$, the order of q. so $\lambda q^{1-j} - \lambda^{-1} q^{j-1} = 0 \Rightarrow \lambda^2 = q^{2(j-1)}$

so $\lambda = \pm q^{(j-1)}$. (another necessary cond. for proper submod.)

Now check that when $b = 0$, $\lambda = \pm q^n$, $0 \leq n < \ell - 1$, then the m_i with $i > n$ are a submodule. (straightforward. in particular $E \cdot m_{n+1} = 0$)

In particular we learn if $0 \leq n < \ell$, recover same structure of simple $\mathcal{U}$-modules $L(n, \pm)$ in $Z_0(\pm q^n)$ (compare two actions)

Also learn that $Z_0(\pm q^n)$ with $0 \leq n < \ell - 1$ can't be semisimple, because the complement of the $L(n, \pm)$ wouldn't have form described above.

In fact, there are no simple finite-dim'l $\mathcal{U}_\zeta$-modules of dimension $> \ell$.

(Kassel, Prop. VI.5.2) $k = \mathbb{C}$ / alg. closed field.

pf sketch: if V: simple fin dim'l, $\dim > \ell$, then do cases:

(a): there is a lowest weight vector: $Kv = \lambda v$, $Fv = 0$.

then $\langle v, Ev, \ldots, E^{\ell-1}v \rangle$ is a submodule (of dim $\leq \ell$)

reason: $E \cdot E^{\ell-1} v = E^\ell v$ and E^ℓ central so acts by scalar on V

(b) No lowest wt. vector. Take v s.t. $K \cdot v = \lambda v$, consider

$\langle v, Fv, \ldots, F^{\ell-1} v \rangle$: similar idea as before.

need to use commutation rel'ns of E, F to check this.