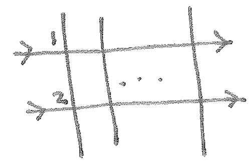
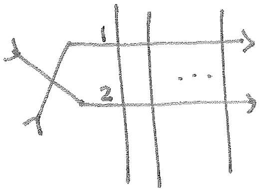
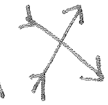
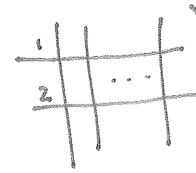
If we have a solution R'' to YBE, how does it prove commutativity of transfer matrices?

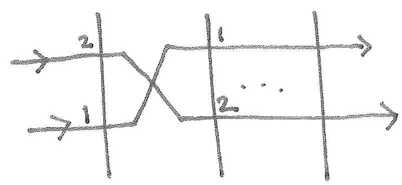
Pictorial proof · Start with $Z\Big($  $\Big)$

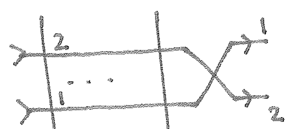
Step
Claim 1.

$Z\Big($  $\Big) = wt\Big($  $\Big) \cdot Z\Big($  $\Big)$

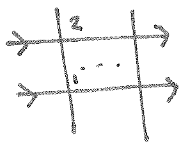
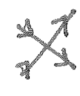
Step 2 ⟶ ||

Use YBE to conclude...

$Z\Big($  $\Big)$ $\overset{\text{YBE}}{\underset{\text{again}}{=}} \cdots \overset{\text{YBE}}{\underset{\text{again}}{=}}$

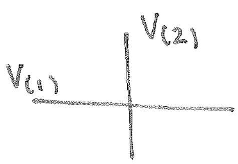
$= Z\Big($  $\Big)$

so these transfer matrices commute!

$= Z\Big($  $\Big)\ wt\Big($  $\Big)$

LECTURE 5 BEGINS...

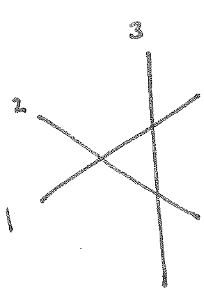
Algebraic interpretation



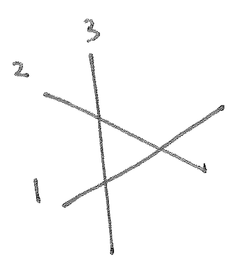
$R \in End(V_{(1)} \otimes V_{(2)})$

vertical strand – one copy of V

horizontal strand – other copy of V.

In 

3 strands.

⟵ coeff. in $End(V \otimes V \otimes V)$ vs 

↓ as endoms

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

in our case: $R''_{12}\, R(\lambda_1)_{13}\, R(\lambda_2)_{23}$

$= R(\lambda_2)_{23}\, R(\lambda_1)_{13}\, R''_{12}$

pictorial pf is given algebraically in 7.5.3 of Chari–Pressley.

Does there exist a solution R'' to QYBE in our 6-vertex model example?

Yes, in fact given $R(\lambda_1), R(\lambda_2)$, then $R'' = R(\lambda_3)$

with $\lambda_3 - \lambda_1 + \lambda_2 + \lambda_1\lambda_2\lambda_3 = (q+q^{-1})\lambda_3\lambda_2$.

Natural to ask for general conditions on 6-vertex weights so that a QYBE exists. $\longrightarrow$ (next page)

Definition: A vertex model is "solvable" (a.k.a. "integrable") if its Boltzmann weights satisfy a QYBE.

exact solns of partition functions $\longleftarrow$ Commuting families of operators

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QYBE $\rightsquigarrow$ commuting transfer matrices $\rightsquigarrow$ closed form solution to partition function of a lattice model

Add to this: Quantum group modules. $\nearrow$ QYBE

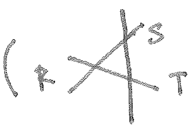
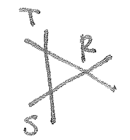
Technical sentence: Quantum groups are examples of "quasitriangular Hopf algebras" $(\underset{\text{Hopf alg.}}{H}, R \in H \otimes H)$ with nice properties, including

on $H \otimes H \otimes H$, $\quad R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad$ (abstract QYBE)

And Given (ρ, V): rep'n of H, then $(\rho \otimes \rho)(R) \in \mathrm{End}(V \otimes V)$ is matrix valued solution of QYBE.

When does a solution to Yang-Baxter equation exist? (6-vertex model)

Given $S, T \in \mathrm{End}(V \otimes V)$, want $R \in \mathrm{End}(V \otimes V)$ s.t.

Z () = Z ()

S has 6 wts. Let $wt_S(SW) = a_1$ $\quad wt_S(NW) = b_1$ $\quad wt_S(EW) = c_1$

$wt_S(NE) = a_2$ $\quad wt_S(SE) = b_2$ $\quad wt_S(NS) = c_2$

define invariants $\Delta_1(S) = \dfrac{a_1a_2 + b_1b_2 - c_1c_2}{2a_1b_1}$.

$$\Delta_2(S) = \frac{a_1a_2 + b_1b_2 - c_1c_2}{2a_2b_2}$$

Thm (Baxter, Brubaker-Bump-Friedberg) ← general case

field-free case $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$

$\Delta_1(S) = \Delta_1(T)$, $\Delta_2(S) = \Delta_2(T)$ $\iff$ $\exists\, R$ s.t. QYBE is solved.

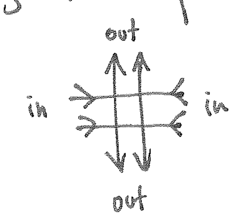
(in fact, we give exact description for weights in solution depending on S, T)

Conclude this week with an example: In ice, energy at every Oxygen site is same. (There are other interesting molecules with hydrogen bonds where this is not the case)

So might as well take all wts $= 1$.

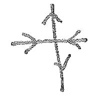
then we're computing Z as the sum of admissible states with fixed boundary condition.

Very special case: $M = N$ (our rectangle is a square)

and boundary is always: 

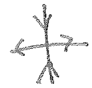
"domain wall boundary conditions"

DWBC

claim : Assign  $\longleftrightarrow$ 1 other 4 $\longleftrightarrow$ 0.

EW

Replace state of ice with a matrix, entries at each vertex

 $\longleftrightarrow$ -1

NS

result will be an $N \times N$ alternating sign matrix.

(that is, matrix of $0, 1, -1$'s where non-zero entries in rows, columns alternate $+1, -1, \ldots, +1$)

in particular, start and end with $+1$.

$N=3$: $\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ is an ASM.

(the only one not a permutation matrix)

exactly one 1 in each row, each column.

To find Z in this case:

How many ASMs are there?

$1, 2, 7, \ldots$?

Conjecture (Mills, Robbins, Rumsey)

$$\# \text{ of } N\times N \text{ ASMs} = \frac{1!\,4!\,7!\ldots(3N-2)!}{N!\,(N+1)!\ldots(2N-1)!}$$

Solution (Kuperberg):

Boltz. weights $= 1$ won't help us in our techniques.

(first proved by Zeilberger - difficult proof establishing bijection with totally symm., self-complem. plane partitions)

Use clever weights with non-trivial YBE.

Let $[x]_q = \dfrac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}}$. Wts are:

EW	NS	SW	NE	SE	NW
$-q^{-x/2}$	$-q^{x/2}$	$[x-1]_q$	$[x-1]_q$	$[x]_q$	$[x]_q$

$R(x)$

Then $R(x)$, $R(y)$ satisfy QYBE w/ solution $R(z)$ where $x = y+z$.

Key point : For $x = [2]_q + 2$,

$Z(n; \frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0)$

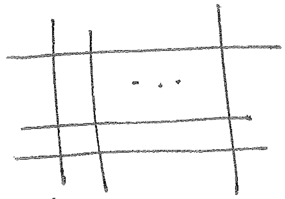
$= (*) \cdot A(n; x)$

explicit const. in q.

refined count. with x^k if $\#(-1)$'s $= k$.

then in

α_0



α_{n-1}

β_0 $\quad$ β_{n-1}

at each vertex $x := \alpha_i - \beta_j$ in pos. i,j.

Solve for $Z(n; \underline{\alpha}, \underline{\beta})$

YBE implies Z symmetric in $\underline{\alpha}$'s, $\underline{\beta}$'s.