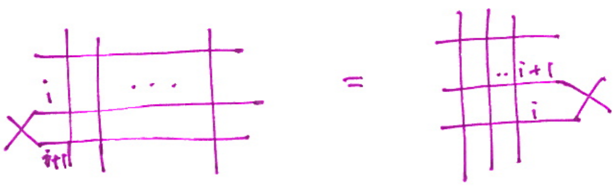
On Monday, saw how QYBE + train argument $\Rightarrow$ commuting transfer matrices

(in pictures: )

For our weights, implied that partition function $Z(\underline{x})$ with $\underline{x} = (x_1, \dots, x_r)$ is symmetric in $\underline{x}$ variables.

params for each row

Let's explore this – which symmetric functions arise as partition functions of lattice models? Start with 6-vertex models.

End of hour, we were trying to cram in last example –

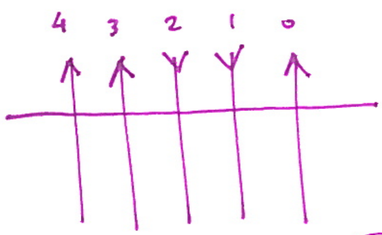
Given integer partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0) \rightsquigarrow \lambda + \rho \quad \rho = \rho_{r-1} = (r-1, \dots, 0)$

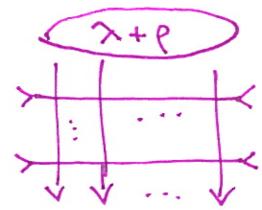
e.x. $\cancel{3}\,2, 2, 0$ is partition of $\cancel{5}\,4$, with 3 parts.
$\lambda = (\lambda_1, \lambda_2, \lambda_3)$

$\rho_3 = (2,1,0)$

then $\lambda + \rho_3 = (4, 3, 0)$

$\rightsquigarrow$ made boundary condition with "up" arrows at $\lambda_i + r - i$, down else.

with corresponding boundary: 

Let $Z_\lambda :=$ partition function of 

(you could add a bunch of $\downarrow$ columns to left and it wouldn't affect much)

i.e. b_1:  : 0
wt of NW.

 : x_i
left @ W

[Insert brief interlude on Schur polys.]

See notes from Lecture #6 ① + ②

$\underline{x}^\rho = x_1^{r-1} \cdots x_r^0$

s_λ: Schur poly.

and Boltzmann weights:

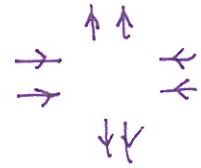
$a_1 = 1, \; a_2 = \cancel{x}_i, \; b_1 = 0, \; b_2 = \cancel{x}_i$

$c_1 = \cancel{x}_i, \; c_2 = 1. \quad (\Delta(z_i) = 0)$

Thm: $Z_\lambda = \underline{x}^\rho \, s_\lambda(\underline{x})$

Last time, giving first results on partition functions.

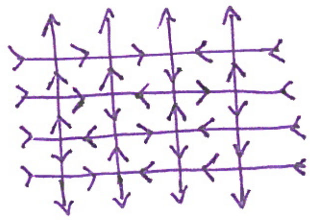
If $M=N$ (square lattice), Boltzmann wts in 6-vertex model, all $=1$,



(domain wall bdy. conditions)

then $$Z_N = \sum_{S \in \text{admissible state}} 1 = \#\text{ of ASMs}$$ (alternating sign matrices)

bijection:



$$\longleftrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

EW: 1, NS: -1, all else: 0.

Sketch pf. of following theorem (Kuperberg): $$Z_N = \frac{1!\,4!\,7!\cdots(3N-2)!}{N!\,(N+1)!\cdots(2N-1)!}$$

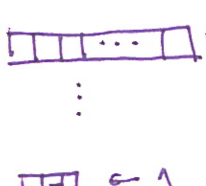
(return to earlier notes pages)

Note: Zeilberger's proof was not bijective. Just showed equinum.

~~Theorem~~ Recall Schur polynomial – very special symmetric function (character values of highest weight representations V_λ of $SL_n(\mathbb{C})$)

combinatorial definition in terms of semi-standard Young tableaux (SSYT)

Given partition $\lambda = (\lambda_1, \dots, \lambda_r)$, make diagram of shape λ



$\leftarrow \lambda_1$ boxes

$\vdots$

$\leftarrow \lambda_r$ boxes

ex: $$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array} = T.$$

for $\lambda = (5,3,1)$

SSYT: filling with alphabet $\{1,\dots,r\}$ s.t. weakly increasing in rows, strictly increasing in columns

$\mathrm{wt}(T) = z_1^{\#1\text{'s}} \cdots z_r^{\#r\text{'s}}$

in our example $\mathrm{wt}(T) = z_1^2 z_2^4 z_3^3$.

$$s_\lambda(\underline{z}) := \sum_{T \in SSYT(\lambda)} \underline{z}^{wt(T)}.$$

Rep'n theory: $\rho_\lambda : GL_n(\mathbb{C}) \longrightarrow \mathrm{End}(V)$

$\cup$

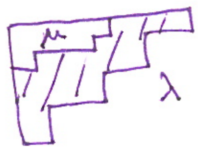
$A \ni \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_r \end{pmatrix}$ diag mats.

$$s_\lambda(\underline{z}) = \mathrm{Tr}\left(\rho_\lambda\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_r \end{pmatrix}\right)$$

Also skew Schur polys $s_{\lambda/\mu}$ where

we fill skew tableaux:

skew shape λ/μ.



$$s_{\lambda/\mu}(\underline{z}) := \sum_{T \in SSYT(\lambda/\mu)} \underline{z}^{wt(T)}$$

Given partition λ (r parts) $\rightsquigarrow$ partition $\lambda+\rho$ with distinct parts

$\rho = (r-1, r-2, \dots, 1, 0)$

$\rightsquigarrow$ row of arrows: up at columns index matches parts of $\lambda+\rho$.

down else.

$\lambda = (2,2,0) \rightsquigarrow \lambda+\rho = (4,3,0)$

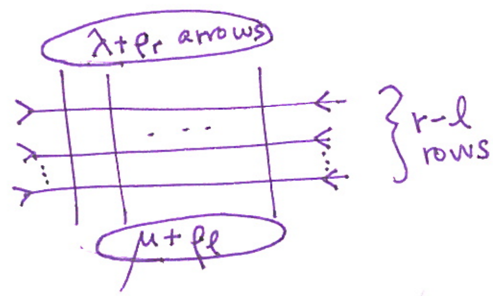
$\rightsquigarrow$ $\begin{array}{ccccc} \uparrow & \uparrow & \downarrow & \downarrow & \uparrow \\ 4 & 3 & 2 & 1 & 0 \end{array}$

(bijection between partitions and rows of up/down arrows)

Given λ, μ, $\mu \le \lambda$, then make lattice

λ: r parts

μ: l parts



(i.e. generic columns, prescribed rows)

$\mu = \phi$, $\lambda = (\underbrace{0, \dots, 0}_{r})$

is Domain Wall Bdy conditions

Thm: (same paper w/ Brubaker-Bump-Friedberg)

2011 - Commun. in Math Physics, $\mu = \phi$ case:

$$Z_{\lambda+\rho} = Z_\rho \cdot s_\lambda\left(\frac{b_2^{(1)}}{a_1^{(1)}}, \dots, \frac{b_2^{(r)}}{a_1^{(r)}}\right)$$

with $Z_\rho = \prod_{i=1}^{r} a_1^{(i)} c_2^{(i)} \cdot \prod_{i<j} \left(a_1^{(j)} a_2^{(i)} + b_1^{(i)} b_2^{(j)}\right)$

for any wts. with $\Delta^{(i)} = 0 \ \forall i$

More complicated for μ gen.: $s_{\lambda/\mu}$

What other symmetric functions are possible?

Know std. symmetric functions $e_r(x)$: sums of degree r monomials with distinct vars

$h_r(x)$ sums of degree r monomials.

s_λ most important (connection to rep. thy)

orthonormal basis w.r.t. symmetric bilinear form $\langle\, ,\rangle$

$m_\lambda(x)$: sums of monomials distinct perms of λ

q,t analogue $\langle\, ,\rangle_{q,t}$ and Macdonald polys. are unique symmetric polys s.t. :

- $P_\lambda(x;q,t)$ are orthogonal w.r.t. $\langle\, ,\rangle_{q,t}$
- and s.t. $P_\lambda(x) = m_\lambda(x) + \sum_{\mu<\lambda} a_{\lambda,\mu} m_\mu(x)$

Hierarchy of symmetric polys.

(non-symmetric Macdonald) $E_\alpha(\underline{x};q,t)$ α : composition

sum over ~~permutations~~ of composition.

$E_\alpha(\underline{x};q,t) \longrightarrow P_\lambda(\underline{x};q,t)$

$P_\lambda(\underline{x};q,t) \xrightarrow{t=q^a,\ q\to 1}$ Jack polys.

$P_\lambda(\underline{x};q,t) \xrightarrow{q=0} P_\lambda(\underline{x};t)$ (Hall-Littlewood)

$P_\lambda(\underline{x};t) \xrightarrow{t=0} s_\lambda(\underline{x})$ (Schur poly)

April, 2019 Borodin-Wheeler "Non-symmetric Macdonald polys. via integrable vertex models"

Two facts (1) if $\underline{x}$ has n vars, arise naturally from $U_q(\hat{sl}_{n+1})$

std. module V : R-matrices from $\mathrm{Sym}^\bullet(V)$.

(2) Use colored paths to get non-symmetric refinement of symmetric poly.

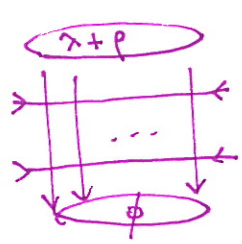
Why is it interesting that $s_\lambda(\underline{x})$ (or any other symmetric function) is representable as a partition function?

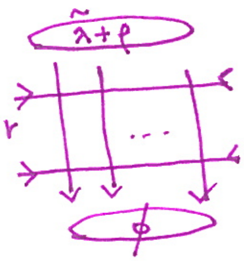
<u>Answer</u>: Lattice models encode so many nice properties.

- branching rules.

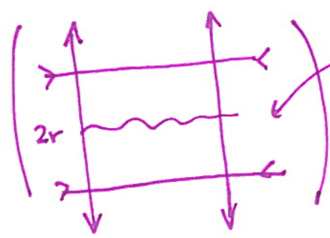
mention interleaving condition on repns of $GL(n, \mathbb{C})$ restricted to $GL(n-1, \mathbb{C})$.

- more clever rules

$\sum_\lambda Z\Big(r$ 

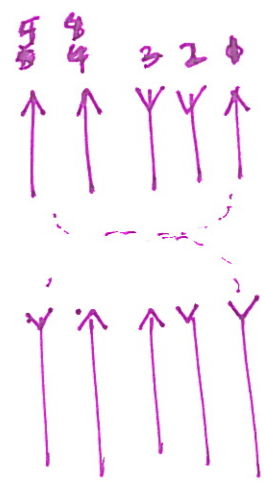
$\Big) \cdot Z\Big(r$ 

$\Big)$

$= Z\Big($ 

$\Big)$

glued them together
get DWBC



$(5,4,1)$
$\downarrow$
$(2,2,0)$
$\downarrow$
$(4,3)$