

An (associative) algebra A / k : field (with unit).

(i.e. ring with action of scalars in k .)

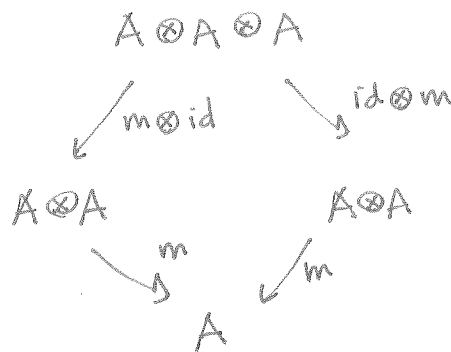
is a vector space with assoc. multiplication map

$$(k\text{-linear assoc. map } A \otimes A \xrightarrow{m} A)$$

and unit element 1_A with $a \cdot 1_A = a = 1_A \cdot a \quad \forall a \in A$.

Write this in diagrams:

(commutative)



For identity diagrams, any $a \in A$ gives map

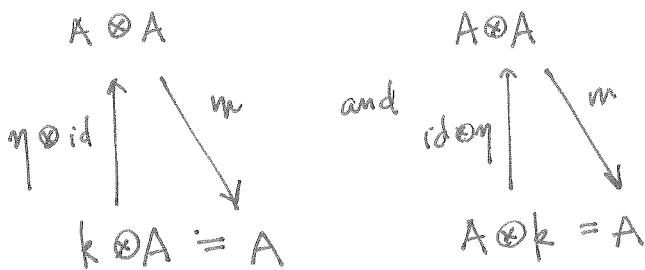
$$\begin{aligned}
 \eta_a: k &\rightarrow A \\
 \lambda &\mapsto \lambda \cdot a
 \end{aligned}$$

$$\text{so } \eta_a(1) = a$$

and for unit map

$$\eta := \eta_{1_A}(1) = 1_A.$$

So existence of a unit 1_A in A is equivalent to map η s.t.



so here $\eta(1) \otimes a = a$

so $\eta(1)$ behaving like left unit for A .

Example: Group algebra kG

(vector space with basis indexed by elements of G ;

call them e_g , with

$$e_g \cdot e_h = e_{g+h} \quad \text{gp. mult.}$$

Example 2: Tensor algebra $T(V)$

$$k \oplus V \oplus V \otimes V \oplus \dots$$

$$= \bigoplus_{i=0}^{\infty} V^{\otimes i}$$

"linear combinations of finite strings of vectors"

then

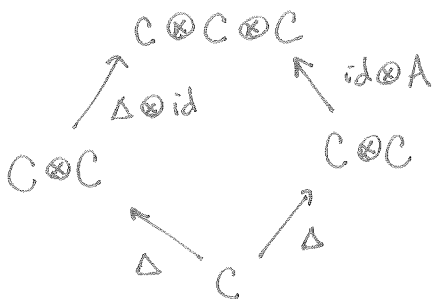
multiplication = concatenation of strings.

Commutative diagrams can be powerful way to make definitions + assertions/proofs. (2)

When we reverse all arrows, get dual objects/theorems.

So co-algebra ~~is~~ C is vector space/ k with comultiplication/coproduct $\Delta: C \rightarrow C \otimes C$
omit hyphen

Which is co-associative



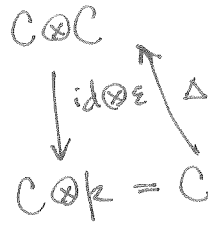
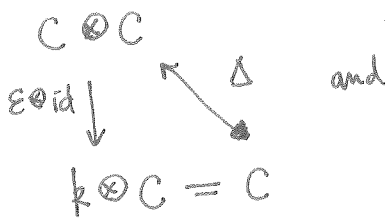
Typical notation uses fact that elts. of $C \otimes C$ are finite linear combinations of symbols $\sum_i c_{i(1)} \otimes c_{i(2)}$

so $\Delta: c \in C \mapsto \sum_i c_{i(1)} \otimes c_{i(2)}$

Bulky: $c \mapsto \sum c_{i(1)} \otimes c_{i(2)}$
 So some write or just drop sum (make it implicit)

And has counit $\epsilon: C \rightarrow k$

with



$\Delta: c \mapsto c_{(1)} \otimes c_{(2)}$

"Sweedler notation"

So want $\Delta(c) = c_{(1)} \otimes c_{(2)}$
 $\downarrow \epsilon \otimes id$
 $1 \otimes c = c.$

Example: Group algebra kG can be endowed with co-alg. structure.

$\Delta: e_g \mapsto e_g \otimes e_g$, $\epsilon(e_g) = 1$ (unit in k)
 $\forall g \in G$ $\forall g \in G.$

(easy check to see diagrams commute above)

How should we write $C \xrightarrow{\Delta} C \otimes C \xrightarrow{\Delta \otimes id} C \otimes C \otimes C$ in Sweedler notation?
 $c \mapsto c_{(1)} \otimes c_{(2)} \mapsto (c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)} \otimes c_{(2)}$

drop these big parens in general.

Given algebras, coalgebras C_1, C_2 , make new algebras $A_1 \otimes A_2, C_1 \otimes C_2$. (3)

- underlying vector space is tensor product of vector spaces

- for $A_1 \otimes A_2$, multiplication map given by: $(a_1 \otimes a_2)(a'_1 \otimes a'_2) = (a_1 a'_1) \otimes (a_2 a'_2)$
 really should write \otimes here too. (mult. in A_1 , mult. in A_2)

— for $C_1 \otimes C_2$, $\Delta: C_1 \otimes C_2 \rightarrow (C_1 \otimes C_2) \otimes (C_1 \otimes C_2)$
 $C_1 \otimes C_2 \mapsto (C_{1(1)} \otimes C_{2(1)}) \otimes (C_{1(2)} \otimes C_{2(2)})$

(better: $\Delta: C_1 \otimes C_2 \xrightarrow{\Delta_1 \otimes \Delta_2} C_1 \otimes C_1 \otimes C_2 \otimes C_2 \xrightarrow{\text{id} \otimes \tau \otimes \text{id}} C_1 \otimes C_2 \otimes C_1 \otimes C_2$
 Δ on $C_1 \otimes C_2$ realized as composition where τ is the "flip map" $C_1 \otimes C_2 \rightarrow C_2 \otimes C_1$.)

What about tensor algebra $T(V)$?

Two coproduct structures are possible:

EASY: $\Delta(v_1 \otimes \dots \otimes v_n) \stackrel{\text{df}}{=} \sum_{j=0}^n (v_1 \otimes \dots \otimes v_j) \otimes (v_{j+1} \otimes \dots \otimes v_n)$
 in $T(V) \otimes T(V)$
 where $v_0 = v_{n+1} = 1 \in k$.

HARD: $\Delta(v) = v \otimes 1 + 1 \otimes v$
 in $T(V) \otimes T(V)$.

Extend, so $\Delta(v_1 \otimes v_2) = \Delta(v_1) \otimes \Delta(v_2)$
 $= (v_1 \otimes 1 + 1 \otimes v_1) \otimes (v_2 \otimes 1 + 1 \otimes v_2)$
 in $T(V) \otimes T(V)$ (need to keep separate)

Which one is better?

$\epsilon(v) = 0 \quad v \in V, \quad \epsilon: \mathbb{K} \rightarrow \mathbb{K}$ in $m \in \mathbb{K}$.
 extend.

"foll"
 $= v_1 v_2 \otimes 1 + v_1 \otimes v_2 + v_2 \otimes v_1 + 1 \otimes v_1 v_2$
 write \otimes in $V \otimes V$ as concat.
 \otimes in $T(V) \otimes T(V)$ remains

A bialgebra H is a coalgebra + algebra in which (Δ, ε) are algebra maps. (or equiv., in which (m, η) are coalg. maps). (4)

In diagram-speak:

$$\begin{array}{ccc}
 \textcircled{1} & H \otimes H & \xrightarrow{m} & H & \xrightarrow{\Delta} & H \otimes H \\
 & \Delta \otimes \Delta \downarrow & & & & \uparrow m \otimes m \\
 & H \otimes H \otimes H \otimes H & \xrightarrow{id \otimes \tau \otimes id} & H \otimes H \otimes H \otimes H & &
 \end{array}$$

- ① $\Delta(m(a,b)) = m(\Delta(a), \Delta(b))$
- ② $\Delta(1_H) = 1_H \otimes 1_H$
- ③ $\varepsilon(m(a,b)) = m(\varepsilon(a), \varepsilon(b))$
- ④ $\varepsilon(1_H) = 1$

etc.

$$\textcircled{2} \quad \begin{array}{ccc}
 k & \xrightarrow{\eta} & H \\
 & \searrow \eta \otimes \eta & \uparrow \Delta \\
 & & H \otimes H
 \end{array}$$

So is the group algebra kG a bialgebra? Yes. For example...

Check ①: $\Delta(m(e_g, e_h)) = \Delta(e_{g*h}) = e_{g*h} \otimes e_{g*h}$

$m(\Delta(e_g), \Delta(e_h)) = m((e_g \otimes e_g) \otimes (e_h \otimes e_h))$. Remember how to do mult. ✓

Does one of the comult. structures on $T(V)$ yield a bialgebra? or both

Yes. the HARD one. Try to verify this yourself.

One last axiom - antipode map $S: H \rightarrow H$ (acts like inversion. Bialgebra - quantum semigrp.)

s.t. $m(S \otimes id) \circ \Delta = \eta \circ \varepsilon$

$m(id \otimes S) \circ \Delta = \eta \circ \varepsilon$.

Hopf Algebra - quantum gp

In diagrams:

$$\begin{array}{ccc}
 H & \xrightarrow{\varepsilon} & k & \xrightarrow{\eta} & H \\
 \Delta \downarrow & & id \otimes S, S \otimes id & & \uparrow m \\
 H \otimes H & \xrightarrow{id \otimes S, S \otimes id} & H \otimes H & &
 \end{array}$$

~ drew 2 diagrams in one here...