An (associative) algebra $A \,/\, k$: field (with unit)

(i.e. ring with action of scalars in k.)

is a vector space with assoc. multiplication map

(k-linear assoc. map $A \otimes A \xrightarrow{m} A$)

· and unit element 1_A with $a \cdot 1_A = a = 1_A \cdot a \;\; \forall a \in A$.

Write this <u>in diagrams</u>:

(commutative)

$$\begin{tikzcd}
 & A \otimes A \otimes A \arrow[dl, "m \otimes id"'] \arrow[dr, "id \otimes m"] & \\
A \otimes A \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\
 & A &
\end{tikzcd}$$

For identity diagrams, any $a \in A$ gives map

$$\eta_a : k \to A$$
$$\lambda \mapsto \lambda \cdot a$$

so $\eta_a(1) = a$

and for unit map

$$\eta := \eta_{1_A}(1) = 1_A.$$

So existence of a unit 1_A in A

is equivalent to map η s.t.

$$\begin{tikzcd}
A \otimes A \arrow[dr, "m"] & \\
k \otimes A = A \arrow[u, "\eta \otimes id"] & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \otimes A \arrow[dr, "m"] & \\
A \otimes k = A \arrow[u, "id \otimes \eta"] & A
\end{tikzcd}$$

$$\begin{tikzcd}
\eta(1) \otimes a \arrow[dr, "m"] & \\
1 \otimes a \arrow[u] & = a
\end{tikzcd}$$

so here

so $\eta(1)$ behaving like left unit for A.

<u>Example</u>: Group algebra kG

(vector space with basis indexed by elements of G, call them e_g, with

$$e_g \cdot e_h = e_{\underbrace{g * h}_{\text{gp. mult.}}}$$)

<u>Example</u> 2: Tensor algebra $T(V)$

$$k \oplus V \oplus V \otimes V \oplus \dots = \bigoplus_{i=0}^{\infty} V^{\otimes i}$$

"linear combinations of finite strings of vectors"

then multiplication = concatenation of strings.

Commutative diagrams can be powerful way to make definitions + assertions/proofs.

When we reverse all arrows, get dual objects/theorems.

So co-algebra C is vector space$/k$ with comultiplication/coproduct $\Delta : C \to C \otimes C$

omit hyphen

which is co-associative

$$\begin{tikzcd} & C\otimes C\otimes C & \\ C\otimes C \arrow[ur, "\Delta\otimes id"] & & C\otimes C \arrow[ul, "id\otimes \Delta"'] \\ & C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] & \end{tikzcd}$$

Typical notation uses fact that elts. of $C \otimes C$ are finite linear combinations of symbols $\sum_i c_{i(1)} \otimes c_{i(2)}$

so $\Delta : c \in C \mapsto \sum_i c_{i(1)} \otimes c_{i(2)}$

Bulky. So some write $c \mapsto \sum c_{(1)} \otimes c_{(2)}$ or just drop sum (make it implicit)

$\Delta : c \mapsto c_{(1)} \otimes c_{(2)}$.

"Sweedler notation"

and has counit $\varepsilon : C \to k$

with

$$\begin{tikzcd} C\otimes C \arrow[d, "\varepsilon\otimes id"'] & \\ k\otimes C = C \arrow[ur, "\Delta"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C\otimes C \arrow[d, "id\otimes\varepsilon"'] \\ C\otimes k = C \arrow[u, bend right, "\Delta"'] \end{tikzcd}$$

So want
$$\begin{aligned} \Delta(c) &= c_{(1)} \otimes c_{(2)} \\ &\big\downarrow \varepsilon\otimes id \\ & 1 \otimes c = c. \end{aligned}$$

Example : Group algebra kG can be endowed with co-alg. structure.

$\Delta : e_g \mapsto e_g \otimes e_g$ $\forall g \in G$, $\varepsilon(e_g) = 1$ (unit in k) $\forall g \in G$.

(easy check to see diagrams commute above)

How should we write $C \xrightarrow{\Delta} C\otimes C \xrightarrow{\Delta\otimes id} C\otimes C\otimes C$ in Sweedler notation?

$c \mapsto c_{(1)} \otimes c_{(2)} \mapsto (c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)} \otimes c_{(2)}$

drop these big parens in general.

Given algebras A_1, A_2, coalgebras C_1, C_2, make new algebras $A_1 \otimes A_2$, $C_1 \otimes C_2$.

- underlying vector space is tensor product of vector spaces

- for $A_1 \otimes A_2$, multiplication map given by:

$$(a_1 \otimes a_2)(a_1' \otimes a_2') = (a_1 a_1' \otimes a_2 a_2')$$

really should write $\otimes$ here too.

mult. in A_1 mult. in A_2

- for $C_1 \otimes C_2$,

$$\Delta : C_1 \otimes C_2 \longrightarrow (C_1 \otimes C_2) \otimes (C_1 \otimes C_2)$$

$$c_1 \otimes c_2 \longmapsto (c_{1(1)} \otimes c_{2(1)}) \otimes (c_{1(2)} \otimes c_{2(2)})$$

(better: Δ on $C_1 \otimes C_2$ realized as composition

$$C_1 \otimes C_2 \xrightarrow{\Delta_1 \otimes \Delta_2} C_1 \otimes C_1 \otimes C_2 \otimes C_2 \xrightarrow{id \otimes \tau \otimes id} C_1 \otimes C_2 \otimes C_1 \otimes C_2$$

where τ is the "flip map" $C_1 \otimes C_2 \longrightarrow C_2 \otimes C_1$.)

—

What about tensor algebra $T(V)$?

Two coproduct structures are possible:

EASY: $$\Delta(v_1 \otimes \cdots \otimes v_k) \overset{def}{=} \sum_{j=0}^{k} (v_0 \otimes \cdots \otimes v_j) \otimes (v_{j+1} \otimes \cdots \otimes v_{k+1})$$

in $T(V) \otimes T(V)$

where $v_0 = v_{k+1} = 1 \in k$.

or

HARD: $$\Delta(v) = v \otimes 1 + 1 \otimes v$$

in $T(V) \otimes T(V)$.

Extend, so $\Delta(v_1 \otimes v_2) = \Delta(v_1) \otimes \Delta(v_2)$

$$= (v_1 \otimes 1 + 1 \otimes v_1) \otimes (v_2 \otimes 1 + 1 \otimes v_2)$$

in $T(V) \otimes T(V)$ (need to keep separate)

"FOIL" $= v_1 v_2 \otimes 1 + v_1 \otimes v_2 + v_2 \otimes v_1 + 1 \otimes v_1 v_2$

write $\otimes$ in $V \otimes V$ as concat.

$\otimes$ in $T(V) \otimes T(V)$ remains

Which one is better?

$\varepsilon(v) = 0$ $v \in V$, $\varepsilon : m \mapsto m$ in $m \in k$.

extend.

A **bialgebra** H is a coalgebra + algebra in which (Δ, ε) are algebra maps. (or equiv., in which (m, η) are coalg. maps).

In diagram-speak:

①
$$\begin{tikzcd} H \otimes H \arrow[r, "m"] \arrow[d, "\Delta \otimes \Delta"'] & H \arrow[r, "\Delta"] & H \otimes H \\ H \otimes H \otimes H \otimes H \arrow[rr, "id \otimes \tau \otimes id"] & & H \otimes H \otimes H \otimes H \arrow[u, "m \otimes m"'] \end{tikzcd}$$

① $\Delta(m(a,b)) = m(\Delta(a), \Delta(b))$

② $\Delta(1_H) = 1_H \otimes 1_H$

③ $\varepsilon(m(a,b)) = m(\varepsilon(a), \varepsilon(b))$

④ $\varepsilon(1_H) = 1$

etc.

②
$$\begin{tikzcd} k \arrow[r, "\eta"] \arrow[dr, "\eta \otimes \eta"'] & H \\ & H \otimes H \arrow[u, "\Delta"'] \end{tikzcd}$$

So is the group algebra kG a bialgebra? Yes. For example...

Check ①: $\Delta(m(e_g, e_h)) = \Delta(e_{g*h}) = e_{g*h} \otimes e_{g*h}$

$m(\Delta(e_g)\Delta(e_h)) = m((e_g \otimes e_g) \otimes (e_h \otimes e_h))$. Remember how to do mult. ✓.

Does one of the comult. structures on $T(V)$ yield a bialgebra? or both

Yes. the HARD one. Try to verify this yourself.

One last axiom - antipode map $S: H \to H$ (acts like inversion. Bialgebra - quantum semigp.)

Hopf Algebra - quantum gp

s.t. $m \circ (S \otimes id) \circ \Delta = \eta \circ \varepsilon$

$m \circ (id \otimes S) \circ \Delta = \eta \circ \varepsilon.$

In diagrams:

$$\begin{tikzcd} H \arrow[r, "\varepsilon"] \arrow[dr, "\Delta"'] & k \arrow[r, "\eta"] & H \\ & H \otimes H \arrow[r, "id \otimes S, S \otimes id"] & H \otimes H \arrow[u, "m"'] \end{tikzcd}$$

drew 2 diagrams in one here...