Localization: In ring A, pick multiplicatively closed subset $S \subseteq A \setminus \{0\}$

Form $A \cdot S^{-1} = \left\{ \frac{a}{s} \mid a \in A, s \in S \right\} / \sim$: usual equivalence relation

generalizes

$\frac{a}{s} \sim \frac{b}{t}$ if $(at - bs) \cdot u = 0$, $u \in S$ (necessarily $\neq 0$)

$\uparrow$

then AS^{-1} has natural ring structure.

If A integral domain, then view $A \cdot S^{-1}$ as subring in K: field of fractions

then $\frac{a}{s} \sim \frac{b}{t}$ means $at - bs = 0$.

Most important special case: $S = A \setminus \mathfrak{p}$

$\mathfrak{p}$: prime ideal of A

(in fact $A \setminus \mathfrak{p}$ mult. closed $\Leftrightarrow$ $\mathfrak{p}$ prime.)

write $A_{\mathfrak{p}}$ for $A \cdot (A \setminus \mathfrak{p})^{-1}$.

In particular the map on ideals of A:

$$\mathfrak{q} \mapsto \mathfrak{q} \cdot A_{\mathfrak{p}} = \left\{ \frac{a}{s} \mid a \in \mathfrak{q}, s \in \underbrace{A \setminus \mathfrak{p}}_{S} \right\}$$

gives a 1-1 correspondence between prime ideals $\mathfrak{q} \subseteq \mathfrak{p}$ and prime ideals of $A_{\mathfrak{p}}$.

Prove slight generalization: A: domain

$$\mathfrak{q} \mapsto \mathfrak{q} S^{-1}$$
$$\mathfrak{f} \cap A \leftarrow \mathfrak{f}$$

gives 1-1 corresp. between prime ideals $\mathfrak{q} \subseteq A \setminus S$ and prime ideals in AS^{-1}

pf: First show if $\mathfrak{q}$ prime, then

$\mathfrak{q} S^{-1} = \left\{ \frac{a}{s} \mid a \in \mathfrak{q}, s \in S \right\}$ is prime in AS^{-1}.

Follow definitions: if $\frac{a_1}{s_1}, \frac{a_2}{s_2}$ s.t. their product $\frac{a_1 a_2}{s_1 s_2} \in \mathfrak{q} S^{-1}$ i.e. $= \frac{b}{s}$

$a_i \in A,\ s_i \in S$ $\qquad b \in \mathfrak{q}, s \in S$

want show either $a_1, a_2 \in \mathfrak{q}$. Then $s\, a_1 a_2 = \underbrace{s_1 s_2 b}_{\in \mathfrak{q}}$ while $s \notin \mathfrak{q}$ since $\mathfrak{q} \subseteq A \setminus S$

$\Rightarrow a_1 a_2 \in \mathfrak{q}$ and since $\mathfrak{q}$ prime a_1 or a_2 in $\mathfrak{q}$.

i.e. $\frac{a_1}{s_1}$ or $\frac{a_2}{s_2}$ in $\mathfrak{q} S^{-1}$.

<u>claim</u>: $\mathfrak{q} = \mathfrak{q} S^{-1} \cap A$ (i.e. map back in other direction gives identity.)

If an elt is in $\mathfrak{q} S^{-1} \cap A$ it has two representations: $\frac{b}{s} = a$ $\quad a \in A, b \in \mathfrak{q}, s \in S$

$\Rightarrow b = sa \in \mathfrak{q} \Rightarrow a \in \mathfrak{q}$ since $\mathfrak{q} \subseteq S \setminus A$. // other containment in claim clear.

for other direction, if f is prime ideal of $A \cdot S^{-1}$

then $\mathfrak{q} = f \cap A$ is clearly prime and further, $\mathfrak{q} \subseteq A \setminus S$

because any $s \in S \cap \mathfrak{q}$ would give $1 = s \cdot \frac{1}{s} \in f$. ↯.

To show $(f \cap A) S^{-1} = f$ (composition is identity), $\subseteq$ clear.

for $\supseteq$, if $\frac{a}{s} \in f$ then $a = \frac{a}{s} \cdot s \in f \cap A \Rightarrow \frac{a}{s} = a \cdot \frac{1}{s} \in (f \cap A) S^{-1}$

Example of when added generality is useful:

$S = A \setminus \bigcup_{i \in X} \mathfrak{p}_i$. Then AS^{-1}: has prime ideals in bijection w/ ~~[illegible]~~ those primes of A contained in X.

In short, we've shown $A_{\wp}$ is "local ring" – i.e. has a unique maximal ideal $\wp \cdot A_{\wp} = \wp S^{-1}$ with $S = A \setminus \wp$. Compare to say, field, where taking fraction field of domain kills all ideals.

Corollary: $\exists$ canonical embedding ϕ:

$$A/\wp \hookrightarrow A_{\wp}/\wp A_{\wp} \quad \text{for any prime } \wp.$$

If $\wp$ is maximal (which occurs when A is a Dedekind domain (all primes are maximal))

then in fact $A/\wp^n \simeq A_{\wp}/\wp^n A_{\wp} \quad \forall\, n \geq 1.$

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pf: Define $\phi_\circ(a \bmod \wp^n) = a \bmod \wp^n \cdot A_{\wp}$ for any $n \geq 1$.

If $n=1$, ϕ is injective since $\wp = \wp A_{\wp} \cap A$ so $A_{\wp}/\wp A_{\wp}$ is the field of fractions for $A/\wp$. Thus we obtain isomorphism if $\wp$ maximal, since $A/\wp$ field.

What about $n \geq 1$ and $\wp$ maximal? The isomorphism will follow from fact that, for any $s \in A \setminus \wp$, $\wp^n + s \cdot A = A$

i.e. $s \bmod \wp^n$ is a unit in $A/\wp^n$. Prove this by induction on n.

$\underline{n=1}$: just maximality of $\wp$ so $\wp + s\cdot A = A$. (as ideals, this sum is just "gcd")

If $A = \wp^{n-1} + s\cdot A \Rightarrow \wp = \wp(\wp^{n-1} + s\cdot A) \subsetneq \underbrace{\wp^n + s\cdot A}_{\text{this contains } s}$

But again by maximality, then $\wp^n + sA$ must be A.

ϕ injective for any n: if $a \in A$ is in $\wp^n A_{\wp}$ then write $a = \frac{b}{s}$ $b \in \wp^n$, $s \in S$ so $s \notin \wp$.

$\Rightarrow as \in \wp^n \Rightarrow a \bmod \wp^n = 0$ in $A/\wp^n$.

ϕ is surjective for any n: if $\frac{a}{s} \in A_{\mathfrak{p}}$, then since by above

$\mathfrak{p}^n + s \cdot A = A$, $\exists\ a'$ with $a \equiv s \cdot a' \ (\mathfrak{p}^n)$, ~~...~~ $\Rightarrow$

$\frac{a}{s} \equiv a' \mod \mathfrak{p}^n \cdot A_{\mathfrak{p}}$ so $\frac{a}{s} \mod \mathfrak{p}^n A_{\mathfrak{p}}$ is in image of ϕ //

Fact: in local ring, every elt. not in maximal ideal is unit.

(since principal ideal gen. by said elt is not contained in the (i.e. any) maximal ideal.)

Proposition: $\mathcal{O}$ Noetherian domain, then $\mathcal{O}$ is Dedekind $\Longleftrightarrow$ $\forall$ primes $\mathfrak{p} \neq 0$ the localizations $\mathcal{O}_{\mathfrak{p}}$ are "discrete valuation rings".

Recall "dvr" (D) is principal ideal domain with unique maximal ($\neq 0$) ideal

This means maximal ideal $\mathfrak{p} = (\pi)$ $\quad \pi \in D$: DVR and π is only prime elt.

$(p \mid ab \Rightarrow p \mid a$ or $p \mid b)$

so every elt. in D is of form $\varepsilon \cdot \pi^n$ for some unit ε, some power n.

and so we can attach a valuation to $D^\times$ recording this power of π.

we can extend to field of fractions of D with additional convention that $v(0) = \infty$.

Then v satisfies $v(ab) = v(a) + v(b)$ and $v(a+b) \geq \min(v(a), v(b))$

strengthening of usual valuation axioms on function $v: K^\times \to \mathbb{Z}$.

(usually just triangle ineq.)

Let's prove proposition using the following lemma: